# The Toom Interface Via Coupling

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#### Abstract

We consider a one dimensional interacting particle system which describes the effective interface dynamics of the two dimensional Toom model at low noise. We prove a number of basic properties of this model. First we consider the dynamics on a finite interval [1,N) and bound the mixing time from above by 2N. Then we consider the model defined on the integers. Because the the interaction range of the rates and the jump sizes can be arbitrarily large, this is a non-Feller process. As a result, we can define the process starting from product Bernoulli measures with density  $p \in (0,1)$ , but not from arbitrary measures. We show that the only possible invariant measures are those product Bernoulli measures, under a modest technical condition. We further show that the unique stationary measure on  $[0,\infty)$  converges to i.i.d. Bernoulli variables when viewed far from 0.

## 1 Introduction

In this paper, we consider an interesting interacting particle system originally introduced in [5] to describe the effective dynamics of the interface between two phases in Toom's<sup>1</sup> Model (also known as the North-East, or North-East-Center, model) in the limit of weak noise. We recall here (see [13] for more details) that Toom's model is a discrete time probabilistic cellular automaton on  $\mathbb{Z}^2$  in which the spin configurations  $\sigma_t \in \{-1, 1\}^{\mathbb{Z}^2}$  are updated according to the rule

$$\sigma_{t+1}(i,j) = \begin{cases} \operatorname{sign} (\sigma_t(i,j+1) + \sigma_t(i+1,j) + \sigma_t(i,j)) & \text{with probability } 1 - p - q \\ +1 & \text{with probability } p \\ -1 & \text{with probability } q. \end{cases}$$

The parameters p, q represent noise in the update scheme. It is remarkable, and important for what follows, that, for p and q small enough, the system has two stationary states, one with mostly +1's and the other with mostly -1's.

<sup>&</sup>lt;sup>1</sup>Pronounce Toom with a long o, not with the English pronunciation of oo

One may impose an interface between these two phases by setting the model up in the third quadrant of  $\mathbb{Z}^2$  and fixing boundary conditions for  $\sigma_t(j,0) = +1$  and  $\sigma_t(0,j) = -1$  for all j < 0 and for all t.

If p=q=0, all "up-right" paths from  $(-\infty,-\infty)$  to (0,0) define stable configurations (with + above and - below the path) for the deterministic dynamics and one may ask how these interfaces fluctuate for  $p, q \neq 0$  but small. Heuristically, one may expect that flips off the line separating the + and - regime die out quickly, and the dynamics of the line is governed by flips on it. For example, a spin-flip at a vertex immediate to the left of long vertical segment of the interface will generally cause further spin-flips at vertices adjacent to the segment of interface below. The net effect on the interface is to shift the part of the segment below that point 1 unit to the left. Encoding vertical edges of the line by +1 and horizontal by -1, the authors of [5] arrive at the following effective description of the dynamics at weak noise. First of all, the up-right paths are encoded by spin configurations  $\sigma := (\sigma(x))_{x \in \mathbb{N}} \in \{-1,1\}^{\mathbb{N}}$  (1 corresponds to a vertical segment of the interface and -1 to a horizontal segment). Second, the dynamics on up-right paths is described by a continuous time Markov chain. Each 1 particle is equipped with an exponential rate  $\lambda_+$  clock, and a -1 particle with a rate  $\lambda_-$  clock. We will assume throughout that  $\lambda_+, \lambda_- > 0$  and that  $\lambda_+ + \lambda_- = 1$ , the latter condition simply fixes of unit of time. When the clock rings for a particle of fixed sign, the particle exchanges positions with the first particle to its right of opposite sign. From now on we will refer to these dynamics as the Toom Interface. We will not return to the two dimensional dynamics. Here and below, we will denote this process by  $\sigma_t := (\sigma_t(x))_{x \in \mathbb{N}}$ .

One remarkable feature of this model is that its restriction to the first N vertices  $\{\pm 1\}^{\{1,\ldots,N\}}$  is itself a Markov chain; the dynamics is the same unless a clock rings for a spin in the last block of constant sign in  $\{1,\ldots,N\}$ . For updates of spins in the last block, the dynamics reduces to single vertex spin flips. In the language of [5], there are no finite-size effects. It is easy to check that the restricted chain is irreducible on  $\{\pm 1\}^{\{1,\ldots,N\}}$  and hence has a unique stationary measure  $\pi_N$ . The sequence of measures  $(\pi_N)_{N\in\mathbb{N}}$  is consistent, and this in turn implies that the full chain has a unique invariant measure on  $\{\pm 1\}^{\mathbb{N}}$ ,  $\pi_{\infty}$ , which restricts to  $\pi_N$  on  $\{\pm 1\}^{\{1,\ldots,N\}}$ .

Very little is understood rigorously regarding the behavior of either  $\pi_{\infty}$  or the process  $\sigma_t$ , though the papers [3, 5, 6] contain a number of interesting conjectures, heuristics and numerics. The first paper on the subject, [5], studied the Markov chain defined above as a model describing fluctuations via kinetic roughening, the height function  $h_x(\sigma_t)$  being defined by  $h_x(\sigma_t) = \sum_{i=1}^x \sigma_t(i)$ . The striking observation there is that if  $\lambda_+ = 1/2$ , the statistical properties of the model cannot be in the class governed by the conventional KPZ equation: in this case the process  $h_x(\sigma_t)$  is distributionally invariant under global spin flip.

One way to understand this at a heuristic level is to follow the work of Kardar, Parisi and Zhang and guess the behavior of  $h(\sigma_t)$  in the appropriate scaling limit. The process

should satisfy the SPDE

$$\partial_t h = \kappa \Delta h + W(t, x) + a(\nabla h)^2 + b(\nabla h)^3 \dots,$$

where W is a space-time white noise and the last set of terms make explicit the possible dependence on the gradient of h. If  $\lambda_+ \neq \lambda_-$ , one concludes that only the quadratic term is relevant using scaling theory [7]. However, if  $\lambda_+ = \frac{1}{2}$ , h and -h are identically distributed, which forces a=0 in any putative scaling limit. The extent to which the third order term is relevant is an intriguing open question. It is marginal in the renormalization group sense, and as such [5, 6] argue against its appearance for the scaling limits of microscopic models. The situation here is analogous to the expected relationship between the scaling limit of the Ising model in 4 dimensions and the putative  $\phi_4^4$  field theory.

The simplest manifestation of the above discussion appears in the study of the variance, under  $\pi_{\infty}$ , of the sum of the first L spins. Numerics, Renormalization group calculations and heuristics [5, 12] suggest that

$$\operatorname{Var}_{\pi_{\infty}}\left(\sum_{x=1}^{L} \sigma_{x}\right) \sim \begin{cases} L^{2/3} & \text{if } \lambda_{+} \neq \frac{1}{2}, \\ L^{1/2} \log^{1/4} L & \text{if } \lambda_{+} = \frac{1}{2}. \end{cases}$$

It might help the reader to consider what this implies about the correlations in this model. If  $\operatorname{Var} \ll L$  then the model must exhibit strong negative correlations to cancel the contribution to the variance coming from the term  $\sum \sigma^2 = L$ .

With this background in mind, our paper constitutes the first rigorous analysis of the Toom interface, though the results fall short of answering the most intriguing questions raised in [5], e.g. the above hypothesis on the variance. (The paper [2] analyzes a similar, but different model). Let us now present our main findings. We recall that the total variation distance between two measures  $\mu, \nu$  on a finite sample space  $\Omega$  is defined as

$$\|\mu - \nu\| := \frac{1}{2} \sum_{\sigma \in \Omega} |\mu(\sigma) - \nu(\sigma)|$$

Abusing notation slightly, we also use  $\sigma_t$  to denote the restriction of the chain to  $\{\pm 1\}^{[N]}$  and let  $\mu_t^{\xi}$  denote the distribution of  $\sigma_t$  when starting from the initial configuration  $\xi \in \{\pm 1\}^{\{1,\dots,N\}}$ . Recall that the mixing time of  $\sigma_t$  is defined as

$$\tau_{\min}(N) := \inf \left\{ t : \max_{\xi} \|\mu_t^{\xi} - \pi\| < \frac{1}{2} \right\}$$

Our first result is as follows.

Theorem 1.1. For all  $N \in \mathbb{N}$ ,

$$\tau_{\rm mix}(N) \le 2N$$

A potentially surprising property of the Toom interface model is that it can be defined on the whole of  $\mathbb{Z}$ . In this case the Bernoulli i.i.d. measures are invariant to the dynamics. In other words, the phenomenon of unusually small variance seems to disappear (notwithstanding that one expects, as in ASEP, to recover small variances when studying certain dynamic observables such as the current across an appropriately chosen space-time characteristic). We wish to understand this disparity better. We will do it in two different directions.

The first direction is to study the behavior of  $\pi_{\infty}$  in the bulk, far to the right of 0. Is it Bernoulli? Note that on  $\mathbb{Z}$  all i.i.d. product Bernoulli measures, with any density p, are invariant to the dynamics. On  $\mathbb{N}$  far from the boundary, the prospective Bernoulli measure is fixed by the condition  $\mathbb{E}_{\pi_{\infty}}[\sigma_x] = p$  (this being dependent on  $\lambda_+$  and  $\lambda_-$ ).

Formally, let  $\tau_x$  be the translation by x i.e. for any spin configuration  $\sigma$ , with domain  $D \subset \mathbb{Z}$ , let  $\tau_x \sigma$  denote the spin configuration with domain D + x defined by  $(\tau_x \sigma)(y) = \sigma(y-x)$ . Denote the induced map on the space of probability measures by  $\tau_x^*$ . Studying the behavior of  $\pi_\infty$  to the far right is thus studying  $\lim_{k\to -\infty} \tau_k^* \pi_\infty$ .

**Theorem 1.2.** Consider  $(\tau_k^*\pi_\infty)_{k\in-\mathbb{N}}$  as a sequence of probability measures on  $\{\pm 1\}^{\mathbb{Z}}$ . Then this sequence converges weakly, as  $k\to-\infty$ , to the i.i.d. Bernoulli measure  $\operatorname{Ber}_p$  with

$$\left(\frac{1-p}{p}\right)^2 = \frac{\lambda_+}{\lambda_-}$$

The second direction is to ask: are there any measures on  $\mathbb{Z}$  invariant to the dynamics other than the i.i.d. Bernoulli measures? We show that none exist, under some conditions which promise that information does not flow too fast from  $-\infty$ . While we failed to construct an "exotic" (i.e. non-i.i.d.) invariant measure, we have no good reason to conjecture such an example does not exist, it seems a condition on the flow really is necessary. As the specific conditions we use are somewhat lengthy to state, we defer the statement of this result, Theorem 2.6, to the next section.

#### 1.1 Proof ideas

The main tool that we employ is a coupling. Let  $\sigma^1$  and  $\sigma^2$  be two starting configurations. We wish to construct a coupling of the Toom processes starting from  $\sigma^i$  which makes them attempt to become similar with time. We perform this coupling as follows. We start with independent Poisson clocks (one for each vertex) each with rate 1. Suppose there is a Poisson arrival at time t and at a site x. We examine  $\sigma^1_t(x)$  and  $\sigma^2_t(x)$ . If  $\sigma^1_t(x) = \sigma^2_t(x)$  we want the particles at x to move together. To obtain the proper particle clocks we have to reduce the rate, so we throw a coin with probability 1/2 and make them both walk if it succeeds (for this informal discussion we assume  $\lambda_+ = \lambda_- = 1/2$ , the  $\lambda_+ \neq \lambda_-$  case is similar). If  $\sigma^1_t(x) \neq \sigma^2_t(x)$  then we again throw a coin with probability 1/2: if it succeeds we make  $\sigma^1$  walk, and if it fails, we make  $\sigma^2$  walk. It is easy to check that both  $\sigma^i_t$  are Toom processes, so this is indeed a coupling.

Let us examine **discrepancies** i.e. x such that  $\sigma^1(x) \neq \sigma^2(x)$ . A Poisson arrival at x will force  $\sigma^1(x) = \sigma^2(x)$  after it, but a discrepancy might form somewhere to the right of x, call this site y. We say that the discrepancy at x moved to y. Our discrepancy might also move because of a Poisson arrival before x, but the key point is that it in all cases it moves to the right. The important points regarding discrepancy dynamics are as follows: discrepancies are never created. When they move, they only move to the right. They may annihilate each other, but only by collisions between opposite types; e.g. a "+discrepancy" (a discrepancy where  $\sigma^1(x) = 1$  and  $\sigma^2(x) = -1$ ) hits a "-discrepancy". Let us note here that this coupling is attractive: If  $\sigma_0^1 \leq \sigma_0^2$  pointwisely, then they remain so for all future time.

Let us sketch how the coupling gives our results.

Sketch of a proof of Theorem 1.1. Recall that we want to show that the mixing time on a finite interval of length N is bounded by 2N. We couple two processes on this finite interval with arbitrary starting configurations and examine the discrepancies. They move right with speed bigger or equal to 1/2 and fall off the right edge. By time 2N they are all gone, and the configurations are the same. This is well-known to imply a mixing time bound.

Sketch of a proof of Theorem 1.2. Recall that we wish to show that the Toom process on  $\mathbb{N}$ , examined at x, is approximately Bernoulli. We couple the half-line process to the full-line process with the correct p. In this cases the coupling may create discrepancies. For, a Poisson arrivals at some non-positive x which moves a particle to some  $y \in \mathbb{N}$  does not have a half-line process counterpart (note that whether a discrepancy is created or not depends also on whether  $\sigma^1(y)$  agrees with  $\sigma^2(y)$  prior to the arrival). We wish to analyze the flow of discrepancies across a half-space deep in the bulk. We therefore move to a version where the coupling is stationary too (we already have that both coupled processes are stationary, but the coupling is not necessarily so). This is done using a more-or-less standard limit process. We then examine the rate at which discrepancies flow past a point  $x \in \mathbb{N}$  (denote it by  $j_x$ ). We show that  $j_x$  is a decreasing function of x, with  $j_x - j_{x+1}$  being exactly the rate of annihilations at x. To prove the result, it is enough to show that show  $j_x$  tends to 0. Since the rate of annihilations must be small, the only way that  $j_x$  cannot tend to 0 is if there are, with positive density, long stretches (in space or in time) of discrepancies of a single sign. However discrepancies can only annihilate in pairs, so long stretches of discrepancies of the same sign correspond to periods of time in which the signed sum of discrepancies across 0 is large. Finally, we show that the latter cannot happen often enough to support a non-zero limit for  $j_x$ .  $\square$ 

Sketch of a proof of Theorem 2.6. The theorem will state that the only stationary measures on  $\mathbb{Z}$  are Bernoulli. To show this, we start with a stationary measure  $\mu$  and couple it to all Bernoulli processes at once (themselves coupled so that for every p < q the Bernoulli-q process is pointwise larger than the Bernoulli-p process). To couple more

than two Toom processes at once, just do as follows: once the site to move is selected, throw a random coin, it it falls on heads move all 1s, and if it falls on tails move all -1s. As in the previous proof sketch, we construct a version where the coupling itself is also stationary. We then show that there cannot be any annihilations in the coupling, as the flow of discrepancies is stationary, and annihilation would cause the set of discrepancies to decrease with time (we need here that the flow is finite and space-bounded, which induces some conditions on our measure  $\mu$ ). This means that, compared to any of the Bernoulli-p measure coupled to it, it is either pointwise bigger than it everywhere, or pointwise smaller. There is, thus, a critical p (possibly random) such that  $\mu$  is Bernoulli-p (perhaps except at one point). From here it is not difficult to conclude that  $\mu$  is a mixture of Bernoulli measures.

Of the three, the most accessible is the proof of Theorem 1.1 appearing in  $\S$  2.3.  $\S$  3 is devoted to a proof of Theorem 2.6 while  $\S$  4 gives a proof of Theorem 1.2. Except for some notation set out at the beginning of  $\S$  3 these latter two sections may be read independently of one another.  $\S$  5 contains a number of lemmas used in both  $\S$  3 and  $\S$  4, notably the existence of a stationary coupling.

Let us also mention a second paper in preparation [4]. In that paper, we prove various functional central limit theorems for additive functionals of local observables, local currents, tagged particles and the like. Combining the results of that paper with the present paper, we are in fact able to derive the bound

$$\operatorname{Var}_{\pi_{\infty}}\left(\sum_{x=1}^{L} \sigma_{x}\right) \lesssim L.$$

Going beyond this bound probably requires a new idea beyond the technology developed here and in [4].

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# 2 The Main Coupling and Dynamics on $\{\pm 1\}^{\mathbb{Z}}$

The heuristic given in the run-up to Theorem 1.2 presupposes that the dynamics may actually be defined on  $\mathbb{Z}$ . This is a nontrivial issue as as the process does not have a finite

interaction range – arbitrarily distant parts of the configuration on the negative axis can influence the local jump rate – and hence the standard Hille-Yosida construction, as outlined e.g. in [9], is not applicable. As far as we know, only a few non-Feller interacting particle systems have been constructed, most of them relying on a monotonicity property that is missing here, see e.g. [11, 8]. But beyond applicability of standard tools, there are serious issues of existence and uniqueness. The process is not defined starting from arbitrary starting conditions in any reasonable sense – how would one go about defining it if the starting conditions are, say, + on the entire negative line? Further, even if for a given starting configuration and collection of Poisson arrivals there exists a version of the process which is defined for all time, it is not clear that such a version is necessarily unique. For example, suppose the configuration is  $\cdots + + - - + + - - \cdots$  and that there is a Poisson arrival at -2n at time 1/n, for all n (with -2n the first in the block of two signs). Then forcing the spin at -4n to jump 2 units and all those at -4n+2 to jump one unit is a legal solution, but so is its opposite. We have no example of an initial configuration where such non-uniqueness occurs with positive probability and constructing such an example should be interesting. To circumvent this we start by requiring uniqueness, which is encoded in Definition 2.1 below. But first some preliminaries.

Rather than thinking of  $\lambda_{\pm}$ -Poisson clocks as being attached to particles, we will consider a sequence of i.i.d. rate one Poisson point processes  $(N_x(t))_{x\in\mathbb{Z}}$  associated with vertices  $x\in\mathbb{Z}$ . Besides these Poisson point processes, the sample space on which our coupling is defined supports a two dimensional array of of i.i.d. uniform [0,1] variables  $(U_{x,j})_{x\in\mathbb{Z},j\in\mathbb{N}}$ . Let  $(\Omega,\mathbb{P};\mathscr{B}_{\Omega})$  denote a probability space which supports all these variables. Define the filtration of sigma algebras  $(\mathscr{F}_t)_{t\in\mathbb{R}^+}$  on  $\mathscr{B}_{\Omega}$  by

$$\mathscr{F}_t = \sigma\left(N_x(s) : s \le t; \ U_{x,k} : k \le N_x(t)\right).$$

Finally, let  $D = D([0,\infty) \to \{\pm 1\}^{\mathbb{Z}})$  be the space of cádlág functions from  $[0,\infty)$  to  $\{\pm 1\}^{\mathbb{Z}}$  where  $\{\pm 1\}^{\mathbb{Z}}$  is equipped with the usual product topology. We equip D with the Skorokhod topology.

**Definition 2.1.** A T-process is a pair  $(\mu, F)$  where  $\mu$  is a probability measure on  $\{\pm 1\}^{\mathbb{Z}}$  and F is a function from  $\{\pm 1\}^{\mathbb{Z}} \times \Omega$  to D which is Borel measurable. F need only be defined  $\mu \times \mathbb{P}$ -almost everywhere. We require  $F(\eta, \omega)(0) = \eta$  and further that  $\{F(\eta, \omega)(t) : t \in [0, T]\}$  be  $\mathscr{F}_T$ -measurable.

Alternatively, a T-process is a D-valued random variable  $\sigma$  such that for some  $(\mu, F)$  as above,  $\mathbb{P}(\sigma \in E) = (\mu \times \mathbb{P})(F^{-1}(E))$ .

We denote  $F_t(\eta, \omega) = F(\eta, \omega)(t)$  and similarly  $\sigma_t$  is the random variable on  $\{\pm 1\}^{\mathbb{Z}}$  given by  $\sigma$  at time t.

Nothing has yet been formalized regarding Toom processes in this definition, so we just called it a "T-process" in anticipation of the Toom model, which enters in the next definition. We remark that  $\sigma$  clearly determines  $\mu$  as  $\sigma_0 = \mu$ , and determines F  $\mu \times$ 

 $\mathbb{P}$ -almost everywhere, which is enough, since F is anyway defined only  $\mu \times \mathbb{P}$ -almost everywhere.

**Definition 2.2.** For  $\sigma \in D$ ,  $\omega \in \Omega$ ,  $t \in [0, \infty)$  and x < y in  $\mathbb{Z}$ , we say that a Toom jump is required for  $\sigma$  at (t, x, y) if the following occurs:

- 1. There is a Poisson arrival at (t,x) i.e.  $N(x,t) = N(x,t^{-}) + 1$ .
- 2. If  $\sigma_x(t^-) = 1$  then  $U_{x,N_x(t)}$  is required to be less than  $\lambda_+$ , otherwise it is required to be bigger than  $\lambda_+$ .
- 3.  $\sigma_x(t^-) = \sigma_{x+1}(t^-) = \cdots = \sigma_{y-1}(t^-) = -\sigma_y(t^-)$ .

A Toom process on  $\mathbb{Z}$  is a T-process  $(\mu, F)$  such that  $\mu \times \mathbb{P}$ -almost surely,  $F(\eta, \omega)$  has the following properties

- 1.  $F_t(\eta,\omega)(x)$ , considered as a function of t, has only finitely many jumps in any finite interval, for any x.
- 2. If a Toom jump is required for  $F(\eta, \omega)$  at (t, x, y), then there are spin flips at x and y at time t. Otherwise there are no jumps at time t.

A Toom process on S for some  $S \subset \mathbb{Z}$  is the same, except that in Clause 2 we also require  $x \in S$ .

We are now in a position to define our coupling, which is simply using the same  $\omega$  to run a number of different Toom processes. Formally,

**Definition 2.3.** Let  $\{(\mu^i, F^i) : i \in I\}$  be two or more Toom processes (not necessarily on the same subset of  $\mathbb{Z}$ ). When we discuss a "coupling of the  $(\mu^i, F^i)$  started from  $\mu$ " we mean the following:  $\mu$  is assumed to be a measure on  $\prod_{i \in I} \{\pm 1\}^{\mathbb{Z}}$  whose marginals are the  $\mu^i$ . The coupling is then the collection of the D-valued random variables  $\sigma^i$  given by

$$\sigma^i = F^i(\eta^i, \omega) \qquad \sigma^i : \left(\prod_{i \in I} \{\pm 1\}^{\mathbb{Z}}\right) \times \Omega \to D.$$

The coupling with independent starting positions is the object given when  $\mu$  is taken to be  $\prod \mu_i$ .

The reader is urged to verify that this definition corresponds to the verbal definition given in  $\S 1.1$  (this has to do with the precise way in which U enters in the definition).

#### 2.1 Statement of Theorem 2.6

Theorem 2.6 is about stationary Toom processes, so let us start by defining those. The definition uses the natural time shifts on  $\Omega$ . Denote them by  $S_t$ , i.e.  $S_t$  as the map which takes  $N_x(\cdot) \mapsto N_x(\cdot + t) - N_x(t)$  and similarly for the U. We can now define

**Definition 2.4.** A T-process  $(\mu, F)$  is stationary if

1. F preserves  $\mu$ , i.e. for any  $t \in (0, \infty)$  and any  $E \subset \{\pm 1\}^{\mathbb{Z}}$  Borel,

$$\int \mathbb{1}_E(F_t(\eta,\omega)) d\mu(\eta) d\mathbb{P}(\omega) = \mu(E).$$

2. F forms a semi group i.e.  $F_s(F_t(\eta,\omega), S_t\omega) = F_{s+t}(\eta,\omega)$  for all  $t, s \in (0,\infty)$ ,  $\mu \times \mathbb{P}$ -almost everywhere.

It is called "stationary on S" for some  $S \subset \mathbb{Z}$  if we only require clause 1 to hold for events E which depend only on S, formally if  $\eta \in E, \eta|_S = \eta'|_S \Rightarrow \eta' \in E$ .

We need one more technical condition.

**Definition 2.5.** A T-process  $(\mu, F)$  is called regular if for every t one can write  $F_t$  as the  $\mu \times \mathbb{P}$ -limit in measure of functions  $F_t^L$  such that  $F_t^L(\cdot, \omega)$  is continuous (as a function from  $\{\pm 1\}^{\mathbb{Z}}$  to itself), for almost all  $\omega$ .

We have no example of a Toom process which is not regular. Heuristically, constructing a non-regular example seems a similar challenge to constructing an example of a non-unique Toom process (recall the discussion on page 7). We will not do it here, but it is possible to formulate very mild conditions of "no flow of information from infinity at finite time" which would ensure that a process is regular. We are not very happy about this condition, but at least, as will be shown in § 2.2, it is very easy to check in concrete cases. It will be used only once, in Lemma 3.6 below.

Finally, let  $l_y(\sigma)$  and  $r_y(\sigma)$  denote the cardinality of the maximal block of spins with the same sign to the left of y, starting from y-1 and respectively to the right of y, starting from y+1, in particular  $l_y, r_y \ge 1$ .

**Theorem 2.6.** Let  $\sigma$  be a stationary, regular Toom process on  $\mathbb{Z}$ . If the integrability condition

$$\sup_{x \in \mathbb{Z}} \mathbb{E}[(l_x)^{1+\epsilon}(\sigma_0)] < \infty,$$

holds for some  $\epsilon > 0$ , then  $\sigma_0$  is a mixture of product Bernoulli measures.

By making additional assumptions on  $\sigma$ , e.g. spatial translation invariance, one may assume weaker moment conditions. However to keep the presentation streamlined, we'll stick with the above hypotheses on all measures encountered below.

## 2.2 Construction of standard processes

There is no problem in constructing a Toom process on a finite interval or on a cycle: There is only one choice of F which is to go over the Poisson arrivals in the interval

one-by-one and apply the rules stated above. Thus F is well-defined unless there are two Poisson arrivals at the same time or infinitely many Poisson arrivals in a finite interval of time. Since both have probability 0, our F is defined  $\mathbb{P}$ -almost everywhere, which, as already mentioned, is enough.

Turning to a discussion of the evolution on  $\{\pm 1\}^{\{1,\dots,N\}}$  viewed as a factor of  $\{\pm 1\}^{\mathbb{N}}$ , we restrict our attention to initial configurations having both infinitely many +'s and infinitely many -'s in  $\mathbb{N}$  (a property which is preserved for all time),  $F_t$  is a Markov chain on  $\{\pm 1\}^{\{1,\dots,N\}}$ . It is straightforward to check that this is an *irreducible* Markov chain on a finite state space, therefore it has a unique invariant measure. (To show irreducibility, note that to get from a configuration  $\eta$  to another,  $\eta'$ , first make the Poisson clocks of all – sites in  $\eta$  ring from right to left, getting to the all + configuration. Then have all – sites in  $\eta'$  ring from left to right, getting to  $\eta'$ ). With this initial measure,  $(\mu, F)$  is stationary on  $\{1,\dots,N\}$  in the sense of Definition 2.4 (the fact that F preserves  $\mu$  and satisfies the and the semigroup property follows from a direct check of the pertinent definitions). The process is regular because it is in fact itself continuous for all  $\omega$  for which it is defined.

Extending the definition to a Toom process on  $\mathbb{N}$  is almost as easy. Let  $F^N$  be the Toom process on  $\{1,\ldots,N\}$  defined in the previous paragraph (recall that the definition assumed that the initial configuration has infinitely many + and infinitely many - in  $\mathbb{N}$ ). Then  $F^N$  are consistent in the sense that for  $x \in \{1,\ldots,N\}$ ,  $F^N_t(\eta,\omega)(x) = F^M_t(\eta,\omega)(x)$ , for all M>N, all t, all  $\eta$  and almost all  $\omega$ . Hence their stationary measures — denote them by  $\mu^N$  — are also consistent. Taking a limit gives a measure  $\mu^\infty$  and a function  $F^\infty$  and it is straightforward to check that  $(\mu^\infty,F^\infty)$  is a Toom process on  $\mathbb{N}$  if only  $\mu^\infty$  gives zero measure to configurations with a tail of a unique sign.

**Lemma 2.7.**  $\mu^{\infty}$  gives zero measure to configurations with a tail of a unique sign.

Proof. Fix N and let  $M\gg N$ . Examine the event in  $\mu_M$  that  $\sigma_0(N)=\sigma_0(N+1)=\cdots=\sigma_0(M)$  and assume for concreteness that the common value is +. The process exits this state with rate at least  $\lambda_+(M-N)$ , as any Poisson arrival in  $\{N,\ldots,M\}$  with an appropriate U will exit this state. However, it is not difficult to check that returning to this state requires at least one Poisson arrival in  $\{1,\ldots,N-1\}$  or in M. So its rate is bounded by N. We get that the probability of this event is bounded above by  $N/(\lambda_+(M-N))$ . Taking  $M\to\infty$  shows that the probability of  $\sigma_0(N)=\sigma(N+1)=\cdots$  in  $\mu_\infty$  is zero. As N was arbitrary, the lemma is proved.

The Toom process just described is regular because it is a limit of  $F_t^N \to F_t$  also  $\mu^{\infty} \times \mathbb{P}$  almost surely. It is also easy to conclude from the stationarity of  $(\mu^N, F^N)$  that  $(\mu^{\infty}, F^{\infty})$  is stationary.

The last result we wish to show here is that i.i.d. Bernoulli-p processes have a corresponding F which makes them into a stationary, regular Toom process on  $\mathbb{Z}$ . This is a known folk result, but it seems it never appeared in writing so we put it here for completeness.

**Lemma 2.8.** There is an  $F: \{\pm 1\}^{\mathbb{Z}} \times \Omega \to D$  such that for all  $p \in (0,1)$  the couple  $(\operatorname{Ber}_p, F)$  is a stationary, regular Toom process.

Let us isolate the first step of the proof as a separate claim.

**Lemma 2.9.** To prove Lemma 2.8 it is enough to construct such an F which has the required properties only for  $t < \epsilon$  for some fixed  $\epsilon$ .

*Proof.* Exchange  $\epsilon$  and  $2\epsilon$ , and call the input of the lemma G, i.e.  $G: \{\pm 1\}^{\mathbb{Z}} \times \Omega \to D$  and it has the required properties only for  $t < 2\epsilon$  ( $t + s < 2\epsilon$  for the property that it forms a semigroup). We form F by repeatedly applying G i.e.

$$F_t(\eta, \omega) = G_t(\eta, \omega)$$

$$F_t(\eta, \omega) = F_{t-\epsilon}(F_{\epsilon}(\eta, \omega), S_{\epsilon}\omega)$$

$$t \le \epsilon$$

$$t > \epsilon$$

where  $S_{\epsilon}$  is the time shifts on  $\Omega$ , as in the previous section. Note that we are using here stationarity:  $F_{t-\epsilon}$  is defined only  $\mu \times \mathbb{P}$ -almost everywhere, so the expression only makes sense because the couple  $(F_{\epsilon}(\eta, \omega), S_{\epsilon}\omega)$  has  $\mu \times \mathbb{P}$  as its law.

It is easy to check that F preserves  $\mu$  and that it is a Toom process on  $\mathbb{Z}$ . To check that F forms a semigroup, assume first that  $t < \epsilon$  and get

$$F_{s}(F_{t}(\eta,\omega), S_{t}\omega) = F_{s-\epsilon}(\underbrace{F_{\epsilon}(F_{t}(\eta,\omega), S_{t}\omega)}_{s_{t}\omega}, S_{\epsilon+t}\omega)$$

$$= F_{s-\epsilon}(F_{t+\epsilon}(\eta,\omega), S_{t+\epsilon}\omega)$$

$$= F_{s-\epsilon}(F_{t}(F_{\epsilon}(\eta,\omega), S_{\epsilon}\omega), S_{t+\epsilon}\omega)$$

$$= F_{s+t-\epsilon}(F_{\epsilon}(\eta,\omega), S_{\epsilon}\omega)$$

$$= F_{s+t}(\eta,\omega).$$

The first equality follows from opening the outer F by its inductive definition, the second is obtained by applying the semigroup property to the inner term (marked by a brace), which is allowed since  $t + \epsilon < 2\epsilon$ . The third is reopening in the opposite order of t and  $\epsilon$ . The fourth is by assuming the semigroup property has been proved inductively for  $s - \epsilon$  and the fifth is again the definition of F. This shows the case  $t < \epsilon$  by induction on s. Concluding the case of general t is similar and we will skip it.

Finally we need to show that F is regular. We show that by induction on t so we will assume it has already been proved for t and will demonstrate it up to  $t + \epsilon$  (our assumption on G is the induction base). Fix  $\delta > 0$  and let  $F_t^L$  satisfy that

$$\mu \times \mathbb{P}\Big(\Big\{(\eta,\omega): d(F_t(\eta,\omega), F_t^L(\eta,\omega)) < \frac{1}{2}\delta\Big\}\Big) > 1 - \frac{1}{3}\delta.$$

Since  $F_t^L(\cdot,\omega)$  is defined on a compact space, it has a modulus of continuity (which might depend on  $\omega$ ). Take such a modulus which holds for all but  $\frac{1}{3}\delta$  probability, i.e. a  $\gamma$  which

satisfies

$$\mathbb{P}(\{\omega: d(\eta, \eta') < \gamma \Rightarrow |F_t^L(\eta, \omega) - F_t^L(\eta', \omega)| < \frac{1}{2}\delta\}) > 1 - \frac{1}{3}\delta.$$

Finally pick M such that

$$\mu \times \mathbb{P}(\{(\eta, \omega) : d(G_{\epsilon}^{M}(\eta, \omega), G_{\epsilon}(\eta, \omega)) < \gamma\}) > 1 - \frac{1}{3}\delta.$$

Thus  $F_t^L(G_{\epsilon}^M(\eta,\omega), S_{\epsilon}\omega)$  is a  $\delta$ -approximation of  $F_t(G_{\epsilon}(\eta,\omega), S_{\epsilon}\omega)$ , which, by the definition of F, is the same as  $F_{t+\epsilon}$ . And of course, it is continuous for almost all  $\omega$ . This finishes the lemma.

Proof of Lemma 2.8. We will construct F by taking the limit of finite systems with periodic boundary conditions. Let us define the system with periodic boundary conditions formally, even though there are no surprises. We define  $\rho = \rho_t^{(L)}$  to be the following process on  $\{\pm 1\}^{(-L,L]}$  given as a function of Poisson arrivals: suppose we have an arrival at time t and position x. Let y be the cyclically first point right of x having opposite sign, i.e.  $\rho_{t-}(y \mod 2L) = -\rho_{t-}(x)$  and  $\rho_{t-}(z \mod 2L) = \rho_{t-}(x)$  for all x < z < y (here and below,  $y \mod 2L$  is the element of (-L, L] congruent to  $y \mod 2L$ ). Now define

$$\rho_t(z) = \begin{cases} \rho_{t-}(y) & z = x \\ \rho_{t-}(x) & z = y \\ \rho_{t-}(z) & \text{otherwise.} \end{cases}$$

(for completeness let us stipulate that if  $\rho_0$  is the configuration with all + or all - then  $\rho_t = \rho_0$  for all t). A simple check shows that for every  $k \in \{0, \dots, 2L\}$ , the measure which is uniform over configurations with exactly k + signs is stationary. Hence so are the Bernoulli-p measures. Denote by  $F^L$  the map  $\{\pm 1\}^{\mathbb{Z}} \times \Omega \to D$  which realizes this process on (-L, L] and freezes the configuration outside the interval.

We will now show that  $F_t^L(\eta, \omega)$  converges  $\operatorname{Ber}_p \times \mathbb{P}$ -almost surely for all p as  $L \to \infty$ . This will construct F, show that it is regular and that it preserves  $\operatorname{Ber}_p$ . By the previous lemma, it is enough to show this claim only up to some small fixed time  $\epsilon$ . We will choose  $\epsilon$  later.

Compare therefore  $F^L(\eta,\omega)$  and  $F^{L+1}(\eta,\omega)$  restricted to some small spatial interval, say [-K,K]. The following is a sufficient (if far from necessary) condition for them to be equal on [-K,K]: there is an  $x\in (-L,-K)$  such that no particle passed over x in either  $F^L$  or in  $F^{L+1}$ . If we show such x exist, the lemma will be proved. We call such x regeneration points.

To show the existence of regeneration points, examine the following parameter

$$I_t := \sum_{i=1-L}^{-K-1} r_i(\rho_t)^4$$

(recall that  $r_i$  is the size of the block of spins to the right of i). Since  $\rho_t$  is Bernoulli, we see that for some C (depending on  $\lambda_+, p$  only),  $I_t \leq CL$  with probability bigger than  $1 - e^{-cL}$  (from here and on we call this "with exponentially large probability", even if c might depend on  $\epsilon$ ). We now claim that also

$$\mathbb{P}\Big(\max_{t\in[0,\epsilon]}I_t\leq CL\Big)>1-e^{-cL}.$$
(1)

To see this, switch to discrete time (i.e. examine a process on a cycle that simply moves one particle at every step) and notice that, again with exponentially large probability, the continuous time interval  $[0, \epsilon]$  corresponds to no more than L steps of the discrete time process. At each step there is exponentially large probability to have  $I_t \leq CL$  and summing over these L times the probability remains exponentially large. This shows (1). Denote the event of (1) by  $\mathscr{G}$ .

Order the Poisson arrivals in the interval (-L, -K) according their time of arrival. Assume the  $k^{\text{th}}$  Poisson arrival caused the particles at x and y to exchange. Define  $A_k = |x - y|$  and let A be the sum of  $A_k$  for all arrivals in  $[0, \epsilon]$ . Under  $\mathscr{G}$  we have

$$\mathbb{E}(A_k^4) \le C$$

and hence A is, with exponentially large probability, bounded by a sum of  $2\epsilon L$  independent variables with bounded  $4^{\rm th}$  moment. This shows that

$$\mathbb{P}(A > C_1 \epsilon L) \le \frac{C}{L^2}.$$

But A bounds the number of non-regeneration points (for  $F^L$ ): Since the same estimate holds also for L+1, we may fix  $\epsilon=1/4C_1$  and get that with probability larger than  $1-CL^{-2}$  a regeneration point may be found. By Borel-Cantelli this means that there exists an  $L_0$  such that for all  $L>L_0$  a regeneration point exists. As discussed above, this proves that F is a regular Toom process that preserves  $\operatorname{Ber}_p$ .

To finish the lemma we need to show the semigroup property for F, up to time  $\epsilon$ . Fix some K sufficiently large such that  $F^L$  has a regeneration point in [K,0) with probability larger than  $1-\delta$  independently of L (as long as L>K). Since  $F^L\to F$  we see that F has a regeneration point in [K,0) with probability larger than  $1-\delta$ . Since  $\delta$  was arbitrary, we get that F has infinitely many regeneration points. But this shows the claim because after a regeneration point we can calculate F as if it were a finite Toom process. Since this satisfies the semigroup property, so does F, and the lemma is proved.

## 2.3 Two simple applications

In this section we give two applications of the coupling. The first, Theorem 1.1, was already stated in the introduction. The second provides exponential decay of correlations

in time. Here is the precise statement:

**Theorem 2.10.** Let f, g be local functions with  $\operatorname{Ber}_p(f) = \operatorname{Ber}_p(g) = 0$  and  $\operatorname{Ber}_p(f^4) = \operatorname{Ber}_p(g^4) = 1$ . Then

$$\mathbb{E}_{\mathrm{Ber}_p}(f(\sigma_0)g(\sigma_t)) \le C\mathrm{e}^{r-ct},\tag{2}$$

with r the length of the smallest interval containing both Supp f and Supp g, and C, c only dependent on  $\lambda_{\pm}$ .

From now on, constants C, c throughout the paper will be allowed to depend on  $p, \lambda_{\pm}$  without further mention.

Proof of Theorem 1.1. Fix  $\phi \in \{\pm 1\}^{\{1,\dots,N\}}$  to be arbitrary, and let  $\psi \in \{\pm 1\}^{\{1,\dots,N\}}$  be distributed according to the stationary measure  $\mu_N$ . Start two Toom processes on  $\{1,\dots,N\}$  from  $\phi$  and  $\psi$  and couple them as above (call the resulting processes  $\sigma_t^{\phi}$  and  $\sigma_t^{\psi}$  respectively). If we show that at some time T that  $\mathbb{P}(\sigma_T^{\phi} = \sigma_T^{\psi}) > \frac{1}{2}$  then T upperbounds the mixing time by definition.

As explained in the introduction, a crucial property of our coupling is that it pushes discrepancies to the right. Let us formalize this statement. Let  $\tau_1$  be the first arrival on site 1 and for all  $j \geq 2$  let  $\tau_j$  be the first arrival on site j after  $\tau_{j-1}$ . Because we are looking at the process with a wall to the left of site 1, once  $\tau_1$  occurs, the value of  $\sigma_t^{\phi}(1) = \sigma^{\psi}(1)$  for all  $t \geq \tau_1$ . By induction, the same is true for all  $\{(j, \tau_j) : j \leq N\}$ . The theorem is proved by observing that  $\tau_N$  is the time it takes for the N'th arrival of a Poisson point process which has rate 1.

Let us remark that there is also a coupling-less version of the argument. Indeed, examining only one process, after  $\tau_1$  the value of  $\sigma_t(1)$  is independent of the initial configuration, and similarly for all  $\tau_j$ . Thus  $\tau_N$  is a forget time and this is equivalent to the mixing time, see [10].

Proof of Theorem 2.10. Take two local functions f, g and, for concreteness, say that  $\operatorname{Supp} f \cup \operatorname{Supp} g \subset [0, r]$ . We define the measure  $\nu$  on  $\sigma = (\sigma^1, \sigma^2) \in (\{\pm 1\}^{\mathbb{Z}})^2$  as follows:

- 1. Both  $\sigma^1$  and  $\sigma^2$  are  $\mathrm{Ber}_p$ -distributed.
- 2. For x < 0,  $\sigma^{1}(x) = \sigma^{2}(x)$ .
- 3.  $\{\sigma^1(x): x \geq 0\}$  is independent of  $\sigma^2$  and idem with  $1 \leftrightarrow 2$ .

Let  $\sigma_t = (\sigma_t^1, \sigma_t^2)$  be the coupling of Toom processes started from  $\nu$  (recall Definition 2.3). Let  $X(\sigma_t)$  denote the position of the "left-most discrepancy" of the configuration  $\sigma_t$ , i.e.  $X(\sigma_t) := \min\{x : \sigma_t^1(x) \neq \sigma_t^2(x)\}$ . We are interested in  $X(\sigma_t)$  for the following reason. Since the support of f lies in  $[0, \infty)$ ,  $f(\sigma_0^1)$  is independent of  $\sigma_0^2$  and therefore of  $\sigma_t^2 \ \forall t \in \mathbb{R}_+$ . Moreover, if  $X(\sigma_t) > r$  then  $g(\sigma_t^1) = g(\sigma_t^2)$ . Since  $f(\sigma_0^1)$  and  $g(\sigma_t^2)$  are independent, we have

$$\mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\sigma_{t}^{1})) = \mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\boxed{\sigma_{t}^{1}})\mathbb{1}\{X(\sigma_{t}) > r\}) + \mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\sigma_{t}^{1})\mathbb{1}\{X(\sigma_{t}) \leq r\}) \\
= \mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\boxed{\sigma_{t}^{2}})\mathbb{1}\{X(\sigma_{t}) > r\}) + \mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\sigma_{t}^{1})\mathbb{1}\{X(\sigma_{t}) \leq r\}) \\
= \mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\sigma_{t}^{2})) - \mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\sigma_{t}^{2})\mathbb{1}\{X(\sigma_{t}) \leq r\}) \\
+ \mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\sigma_{t}^{1})\mathbb{1}\{X(\sigma_{t}) \leq r\}).$$

The first term is in the final equality is simply 0 (by the independence explained above), so we get

$$|\mathbb{E}_{\nu}(f(\sigma_{0}^{1})g(\sigma_{t}^{1}))| \leq 2\mathbb{E}_{\mathrm{Ber}_{p}}\left(f(\sigma_{0}^{1})^{4}\right)^{1/4}\mathbb{E}_{\mathrm{Ber}_{p}}\left(g(\sigma_{0}^{1})^{4}\right)^{1/4}\mathbb{P}(X(\sigma_{t}) \leq r)^{1/2} \leq 2\mathbb{P}(X(\sigma_{t}) \leq r)^{1/2}$$

where we used Cauchy-Schwarz twice for each term, the invariance of  $\operatorname{Ber}_p$  and finally that  $\operatorname{Ber}_p(f^4) = \operatorname{Ber}_p(g^4) = 1$ .

Finally, observe that  $X(\sigma_s)$  is naturally coupled to a Poisson process N(s) which has rate  $\min(\lambda_{\pm})$  so that

$$X(\sigma_t) \ge N(t)$$

The theorem now follows from a large deviation estimate on the Poisson process N(t).  $\square$ 

### 2.4 Generators, Local Rates and Derivatives

One unfortunate consequence of the fact that the Toom interface is non-Feller is that intuitive reasoning involving generators and locally defined rates need to be checked rigorously. Here we state two results which justify our pervasive use of this language throughout the text, as we will almost always assume Condition A below. Because we feel these statements are more of technical rather than actual interest, their proofs are relegated to the end of the paper (see  $\S$  5.3)

We will define our operators on the space of continuous functions from  $\{\pm 1\}^S$ ,  $S \subset \mathbb{Z}$ , to  $\mathbb{R}$ . We start with the flip operator  $F_x$  at site x, defined by

$$F_x f(\sigma) = f(\sigma^x)$$
  $\sigma^x(y) = \begin{cases} \sigma(y) & y \neq x \\ -\sigma(y) & y = x. \end{cases}$ 

For a finite subset  $S \subset \mathbb{Z}$  and  $\tilde{\sigma} \in \{-1,1\}^S$ , we have the indicators

$$\chi_S^{\tilde{\sigma}} = \chi[\sigma(x) = \tilde{\sigma}(x) \, \forall x \in S]$$

and whenever  $\tilde{\sigma}$  is all 1 or all -1, then we simply write  $\chi_S^+$  and  $\chi_S^-$ . We also need the associated projections –

$$P_S^{\tilde{\sigma}}f(\sigma) = \chi_S^{\tilde{\sigma}}(\sigma)f(\sigma).$$

Then the generator of the process is formally defined as

$$\mathcal{L} = \sum_{x < y} \mathcal{L}_{x,y} = \sum_{x < y} (\lambda_+ P_{[x,y-1]}^+ P_y^- + \lambda_- P_{[x,y-1]}^- P_y^+) (F_x F_y - 1),$$

We call this definition formal because  $\mathscr{L}f$  is in general not continuous, due to the infinite sum over x. We need a condition on moments that will almost always be assumed:

Condition A. We say that a Toom process  $\sigma$  satisfies the uniform Condition A if for some  $\eta > 0$ 

$$\sup_{t,x} \mathbb{E}(l_x^{1+\eta}(\sigma_t)) < \infty.$$

We say that it satisfies the local condition A if

$$\sup_{t} \mathbb{E}(l_x^{1+\eta}(\sigma_t)) < \infty \qquad \forall x.$$

If you are reading the online version and ever forget what is Condition A, clicking the letter A should send you to the definition.

**Lemma 2.11.** Let  $\sigma$  be a Toom process satisfying the local Condition A. Let f be a local function. Then for almost all  $\sigma_0$  we have that  $t \mapsto \mathbb{E}(f(\sigma_t) \mid \sigma_0)$  is differentiable in t, the sum defining  $(\mathcal{L}f)(\sigma_0)$  converges, and

$$\frac{d}{dt}\mathbb{E}(f(\sigma_t)\,|\,\sigma_0) = (\mathcal{L}f)(\sigma_0) \tag{3}$$

Further, we have an averaged version,

$$\frac{d}{dt}\mathbb{E}(f(\sigma_t)) = \mathbb{E}((\mathscr{L}f)(\sigma_0)). \tag{4}$$

The same result also holds for a coupling of finitely many Toom processes  $\sigma^i$ : if all of the  $\sigma^i$  satisfy the local Condition A, then the differentiability of local functions also holds for the joint process, with a corresponding formal generator. The proof of this claim follows the proof of Lemma 2.11 verbatim.

Throughout the paper we will also use statements that are slight generalizations of the above. The most prominent example comes in Definition 3.2 and Lemma 3.3 where we handle "instantaneous rates" that should be justified similarly to the rates of change of  $f(\sigma_t)$  above. We will henceforth use reasoning involving rates and currents without explicit justification.

## 3 Invariant Measures on $\mathbb{Z}$

In this section we investigate invariant measures for the Toom interface on  $\mathbb{Z}$ , using the coupling from § 2. Throughout the section we assume that  $\sigma = (\sigma^1, \sigma^2)$  is a pair of Toom processes on  $\mathbb{Z}$  coupled as in Definition 2.3.

We formally define discrepancies. Let

$$\mathbf{D}^{\eta} = \mathbf{D}^{\eta}_{\sigma} := \{ x \in \mathbb{Z} : \sigma^{1}(x) = \eta, \ \sigma^{2}(x) = -\eta \} \qquad \eta = \pm 1,$$
$$\mathbf{D} = \mathbf{D}_{\sigma} := \mathbf{D}^{+}_{\sigma} \cup \mathbf{D}^{-}_{\sigma} = \{ x \in \mathbb{Z} : \sigma^{1}(x) \neq \sigma^{2}(x) \}$$

Rather than focusing on the discrepancies themselves, it is useful to restrict attention to the study of gaps between consecutive discrepancies of type (+,-) and type (-,+), i.e. elements of  $\mathbf{D}^+$  and  $\mathbf{D}^-$ , respectively. Let us, arbitrarily, refer to the first type discrepancies as having signature/sign + and the second type of discrepancies as having signature –. To keep track of "interfaces" between the two types of discrepancy, let, for  $x \in \mathbf{D}^{\eta}$ ,

$$b(x) = \inf\{y > x : y \in \mathbf{D}^{-\eta}\}\$$

should such a y exist and set  $b(x) = \infty$  otherwise. The set of interface discrepancies is then

$$B = B(\sigma) := \{x \in \mathbf{D} : b(x) < \infty \text{ and } (x, b(x)) \cap \mathbf{D} = \emptyset\}.$$

The following lemma is a main ingredient of Theorem 2.6, and the place where the integrability condition is used.

**Lemma 3.1.** Assume  $(\sigma^1, \sigma^2)$  is a stationary coupling of Toom processes which satisfies

$$\sup_{x} \mathbb{E}\left(\max\{l_x(\sigma^1), l_x(\sigma^2)\} \min\{r_x(\sigma^1)r_x(\sigma^2)\}\right)^{1+\epsilon} < \infty.$$

Then

$$\mathbb{P}(\sigma^1 \leq \sigma^2 \ or \ \sigma^2 \leq \sigma^1) = 1$$

Here and below, " $\leq$ " stands for pointwise inequality for all  $t \in [0, \infty)$  and all  $x \in \mathbb{Z}$ . We stress that the phrase " $(\sigma^1, \sigma^2)$  is a stationary coupling" means not only that each one is a stationary Toom process, but also that the coupling is stationary. Formally, let  $\nu$  be the measure on  $(\{\pm 1\}^{\mathbb{Z}})^2$  and  $F^i: \{\pm 1\}^{\mathbb{Z}} \to D$  be the maps that define  $\sigma$  i.e

$$\mathbb{P}(\sigma \in E) = \int \mathbb{1}_E \left( F^1(\eta^1, \omega), F^2(\eta^2, \omega) \right) d\nu(\eta^1, \eta^2) d\mathbb{P}(\omega).$$

Stationarity here refers to the requirement that  $(F_t^1(\eta^1,\omega), F_t^2(\eta^2,\omega))$  is distributed according to  $\nu$  for all t.

Proof of Lemma 3.1. Denote for brevity

$$\max_{x} = \max\{l_x(\sigma^1), l_x(\sigma^2)\}$$
  $\min_{x} = \min\{r_x(\sigma^1), r_x(\sigma^2)\}.$ 

In general, our goal is to show that  $\mathbb{E}[|B|] = 0$  for  $\nu$  the measure on  $(\{\pm 1\}^{\mathbb{Z}})^2$  induced by  $\sigma_0$ . Let us first sketch the proof in case  $\nu$  is translation invariant. We fix an interval I. By stationarity, we are tempted to write

$$0 = \partial_t \mathbb{E}_{\nu}[|B \cap I|] \le \mathbb{E}[\max_{I \in I}] - \min(\lambda_{\pm}) \mathbb{E}[|B^1 \cap I|], \tag{5}$$

where  $B^1 \subset B$  is the set of interface discrepancies that can be annihilated in one step. The inequality in (5) is due to the observations that the first term bounds from above the flow rate of of discrepancies from  $(-\infty, \min I - 1]$  into I while the second term bounds from below the annihilation rate inside I. By hypothesis,  $\sup_x \mathbb{E}[\max I_x] < \infty$  so that these two inequalities together imply that  $\mathbb{E}[|B^1 \cap I|]$  is uniformly bounded in I. Under the assumption of translation invariance of  $\nu$ , this implies  $\mathbb{E}[|B^1|] = 0$ . This argument can then be iterated (considering the discrepancies that can be promoted into  $B^1$  in one step, etc.). Eventually one then concludes that  $\mathbb{E}[|B|] = 0$ . Note that this argument is slightly formal due to the fact that  $|B \cap I|$  is not a local function and therefore the inequality in (5) would need additional justification. Instead of remedying this directly, we proceed to Lemma 3.4, which we need for the general proof of Lemma 3.1 and which immediately settles the translation invariant case. We will need the following definition together with a justifying lemma, here and in §4.

**Definition 3.2.** Let  $\sigma^1$  and  $\sigma^2$  be two coupled Toom processes, and let  $x \in \mathbb{Z}$ . We define  $j_x(\sigma_0)$  to be the infinitesimal rate at which a discrepancy jumps from  $(-\infty, x)$  to  $[x, \infty)$  (we include also in  $j_x$  also the case that the discrepancy annihilates in  $[x, \infty)$  in the same step). Formally, for y < x,  $i \in \{1, 2\}$  and t > 0 we let  $J_{y,t,i}$  be the event that  $\sigma^i$  had a Toom jump at time t at position y, and that 1) for all  $z \in [y, x)$   $\sigma^i_{t-}(z) = \sigma^i_{t-}(y)$  and 2) for some  $z \in [y, x)$ ,  $\sigma^1_{t-}(z) \neq \sigma^2_{t-}(z)$ . Then we define

$$j_x(\sigma_0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{P}(\exists y \le x, t \le \epsilon \text{ and } i \in \{1, 2\} \text{ such that } J_{y,t,i}).$$

We define  $a_x(\sigma_0)$  as the infinitesimal rate of annihilation of discrepancies at x. Formally,

$$a_x(\sigma_0) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{P}(\exists y \le x, t \le \epsilon \text{ and } i \in \{1, 2\} \text{ s.t. } J_{y,t,i} \text{ and } \sigma_{t^-}^i(y) = \sigma_{t^-}^{3-i}(x) \ne \sigma_{t^-}^i(x)).$$

Finally define

$$j_x = \mathbb{E}(j_x(\sigma_0))$$
  $a_x = \mathbb{E}(a_x(\sigma_0)).$ 

These rates are well-defined, as we state now.

**Lemma 3.3.** If  $\sigma^1$  and  $\sigma^2$  are two Toom processes satisfying the local Condition A, then for any coupling of  $\sigma^1$  and  $\sigma^2$ ,  $j_x(\sigma)$  and  $a_x(\sigma)$  are well-defined almost surely and their averages  $a_x, j_x$  are finite. Further,  $j_x$  and  $a_x$  (which have been defined above as the expected quenched rates) are equal to the corresponding annealed rates, i.e.

$$j_x = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}(\#\{discrepancy \ jumps \ from \ (-\infty, x) \ to \ [x, \infty) \ before \ time \ \epsilon\})$$

$$a_x = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}(\#\{annihilations \ at \ x \ before \ time \ \epsilon\}).$$

*Proof.* These claims are variations on Lemma 2.11 as  $a_x(\sigma), j_x(\sigma)$  are defined as conditional time-derivatives. The only difference with Lemma 2.11 is that these derivatives are not naturally equal to  $\mathbb{E}_{\sigma}(\mathcal{L}'f)$  with  $\mathcal{L}'$  the formal generator of the coupled process, but the proof of Lemma 2.11 (given below) applies here as well. For concreteness, let us give the expression for  $a_x(\sigma)$ 

$$a_y(\sigma) = \sum_{x < y} \sum_{\eta = \pm} \sum_{i=1}^2 \lambda_{\eta} \mathbb{1} \{ \sigma \in V_{x,y,i,\eta} \}$$

where  $V_{x,y,i,\eta}$  is the event that  $\eta = \sigma^i(x) = \sigma^i(x+1) = \cdots = \sigma^i(y-1) = -\sigma^i(y)$  while  $\sigma^{3-i}$  satisfies that  $\sigma^{3-i}(y) = \eta$  and for some  $z \in [x,y)$  we have  $\sigma^{3-i}(z) = -\eta$ .

We now pick up the threads of the proof of Lemma 3.1, starting with

**Lemma 3.4.** Let  $\nu$  be the initial measure of a stationary coupling and assume the local Condition A. Then

$$\lim_{|I| \to \infty} \sup_{|I| \to \infty} \frac{1}{|I|} \mathbb{E}\left[|B \cap I|\right] = 0. \tag{6}$$

*Proof.* Given a discrete interval [x, y], define the event

$$E_{x,y}(\sigma) := \{ x \in B(\sigma), \ y = b(x) \}$$

In words,  $E_{x,y}$  denotes the event that there is a boundary discrepancy at x and that the first discrepancy to its right occurs at y. Recall that  $a_y$  is the rate of annihilations at y and that  $j_x$  is the rate of discrepancy flow through x defined above. We now claim that

$$j_x - j_{x+1} = a_x. (7)$$

Indeed, if we denote by  $H_x(t)$  the number of discrepancies that crossed x in the time-interval [0, t], and  $A_x(t)$  the number of annihilations at x in [0, t], then we have

$$H_{x+1}(t) - H_x(t) = A_x(t) + \mathbb{1}(x \in \mathbf{D}(\sigma_0)) - \mathbb{1}(x \in \mathbf{D}(\sigma_t))$$

Taking the expectation, we can drop the two rightmost terms by stationarity. Dividing then by t and taking  $t \to 0$ , we obtain (7) by using Lemma 3.4, more precisely the identification of  $a_x$  as "annealed rates". In particular, since the  $j_x$  are finite and the  $a_x$  nonnegative, we have  $\sum a_x \leq C$ .

Then Lemma 3.4 is a consequence of the following claim.

**Lemma 3.5.** With  $\nu$  as in Lemma 3.4, and for any x and y,

$$\sum_{z=x}^{y} a_z \ge c(y-x)\nu(E_{x,y}). \tag{8}$$

Let us finish the proof of Lemma 3.4 and then attend to Lemma 3.5. Let I = [y, z] be our interval, and let k be some parameter. We get from (8)

$$\sum_{x=y}^{z} \nu(x \in B, \ b(x) - x \le k) \le \sum_{x=y}^{z+k} \frac{k^2}{c(k)} a_x \le C(k). \tag{9}$$

On the other hand, for any k,

$$\sum_{x=y}^{z} \nu(x \in B, \ b(x) - x \ge k) \le \frac{|I| + k}{k}.$$

Lemma 3.4 thus follows.

Proof of Lemma 3.5. The main observations we make are that Toom jumps due to arrivals at sites z > y do not harm us — they cannot move the discrepancy at y — while arrivals at sites z < x only help us — they can push the discrepancy at x closer to or on top of y (such that an annihilation occurs), or can annihilate the discrepancy where it stands, but cannot push it beyond y. To arrive at an annihilation event, it suffices to have at least y - x Poisson arrivals at the location of the discrepancy which is at x at time 0 before any occur between the location of the discrepancy and y.

To use this we examine the behavior in the time interval [0,2]. We get

$$\nu(E_{x,y}) = \mathbb{E}\left(\int_0^1 \mathbb{1}_{E_{x,y}}(\sigma_t) \, \mathrm{d}t\right)$$

$$\leq \mathbb{P}(\exists t \in [0,1] \text{ s.t. } \mathbb{1}_{E_{x,y}}(\sigma_t))$$

$$\leq C(y-x)\mathbb{E}(\#\{\text{annihilations in } [x,y] \text{ before time } 2\})$$

$$= 2C(y-x)\sum_{z=x}^y a_z$$

where the last equality follows from the fact that  $a_z$  is also the annealed rate of annihila-

tions (see the "further" clause of Lemma 3.3). This shows Lemma 3.5 and consequently also Lemma 3.4.

We go back to the proof of Lemma 3.1. Let

$$B_x = \{ y \in B : b(y) \le x \},\$$

and note that  $\mathbb{1}\{y \in B_x\}$  is a local function (its support is [y,x]) in contrast to  $\mathbb{1}\{y \in B\}$ , so we can use Lemma 2.11. In what follows we fix some  $x \in \mathbb{Z}$  and some  $h, k \in \mathbb{N}$ , and we omit all three from the notation to avoid clutter. Set

$$\theta(y) = \begin{cases} 1 & \text{for } y \in [x - h, x], \\ 0 & \text{for } y \in (-\infty, -x - h - k], \\ 1 - \frac{j}{k} & \text{for } y = -x - h - j, 0 \le j \le k. \end{cases}$$

By stationarity

$$0 = \partial_t \mathbb{E} \Big[ \sum_{y=x-h-k}^x \theta(y) \mathbb{1} \{ y \in B_x \} \Big]$$

Let us examine the events which change the value of the function between  $[\cdot]$  in time. The value is decreased by annihilations and by discrepancies leaving the set  $B_x$  (an annihilation might make the discrepancy just before it become a boundary discrepancy, i.e. an element of  $B_x$ , but since  $\theta$  is increasing on  $(-\infty, x]$  the sum still decreases). The latter happens, for a discrepancy at y, when b(y) leaves  $(\infty, x]$ . We ignore annihilations and define

$$X := \min(\lambda_{\pm}) \mathbb{1} \{ \exists y \in B_x \cap [x - h + 1, x] : b(y) \text{ can leave } (-\infty, x] \text{ in one step} \}.$$

Next we examine events which cause  $\sum \theta(y)\mathbb{1}\{y \in B_x\}$  to increase. Examine one boundary discrepancy y. If  $r_y(\sigma^1) \neq r_y(\sigma^2)$  then, because y is a boundary discrepancy, this means y is separated from b(y) by a stretch of equal signs. Thus, a Poisson arrival at y either annihilates with the one of opposite type at b(y) or moves one step to the right. Similarly, a Poisson arrival in the interval of length  $[y - \max_y, y)$  might cause the discrepancy at y to move 1 step to the right or to be annihilated. The sum in these cases increases by no more than  $\theta(y+1) - \theta(y)$ . In the case that  $r_y(\sigma^1) = r_y(\sigma^2)$  the discrepancy at y might move by this common value. Therefore, in all cases we may bound the increase in the sum by  $\theta(y + \min_y) - \theta(y)$ . Defining

$$Z(y) := [\theta(y + \min_{y}) - \theta(y)] \max_{y} \mathbb{1}\{y \in B_x\},\$$

a convenient way to express the above bounds is to say

$$0 \le \sum_{y \le x - h} \mathbb{E}_{\nu}[Z(y)] - \mathbb{E}_{\nu}X. \tag{10}$$

(note that we use here the existence of the annealed generator, recall (4)). To exploit (10), we bound  $\sum_y \mathbb{E}\left[Z(y)\right]$  by splitting the sum over y. For  $x-h-k \leq y \leq x-h$ , we use  $|\theta(y+\min_y)-\theta(y)| \leq \min_y /k$  and Hölder's inequality (for any  $1 with <math>\frac{1}{p}+\frac{1}{q}=1$ ) to get

$$\sum_{y=x-h-k}^{x-h} \mathbb{E}\big[|Z(y)|\big] \leq \left(\frac{1}{k} \sum_{y=x-h-k}^{x-h} \mathbb{E}_{\nu}[(\operatorname{maxl}_{y} \operatorname{minr}_{y})^{q}]\right)^{\frac{1}{q}} \left(\frac{1}{k} \sum_{y=x-h-k}^{x-h} \nu(y \in B_{x})\right)^{\frac{1}{p}}$$

If we choose q sufficiently close to 1, then the first factor is bounded by C (uniformly in h, k, x) by the integrability assumption we placed on  $\nu$ . The second factor decays as  $k \to \infty$  (uniformly in h, x) by Lemma 3.4 and the fact that  $B_x \cap I \subset B \cap I$ . For y < x - h - k, we write

$$\sum_{y < x - h - k} Z(y) \le \frac{1}{k} \max_{x - h - k} \min_{x - h - k}$$

since there can be no more than one boundary discrepancy for which  $\theta(y+\min y) - \theta(y) > 0$ . We conclude that

$$\limsup_{k \to \infty} \sum_{y \le x} \mathbb{E}\left[Z(y)\right] = 0.$$

Combining with (10), we conclude that  $\mathbb{E}_{\nu}[X] = 0$  (recall that X is nonnegative and independent of k). Since this holds for all x, h, it follows that  $\mathbb{E}[|B|] = 0$ .

At this point we have amassed information about stationary couplings of measures but haven't actually shown any example of such a coupling. The latter issue is the subject of the next lemma.

**Lemma 3.6.** Let  $(\mu^1, F^1)$  and  $(\mu^2, F^2)$  be two stationary, regular Toom processes. Let  $\nu$  be any measure on  $(\{\pm 1\}^{\mathbb{Z}})^2$  which has  $\mu^1$  and  $\mu^2$  as its marginals. Let  $\nu_t$  be the result of applying the coupling to  $\nu$  for time t, i.e.

$$\nu_t(E) = \int \mathbb{1}_E((F^1(\eta^1, \omega)(t), F^2(\eta^2, \omega)(t)) \, d\nu(\eta^1, \eta^2) \, d\mathbb{P}(\omega).$$

Then any subsequential weak\*-limit of  $\frac{1}{T}\int_0^T \nu_t$  is a stationary coupling of  $(\mu^1, F^1)$  and  $(\mu^2, F^2)$ .

As this lemma is technical in nature we postpone its proof to § 5.2. The lemma will

also be used in  $\S$  4 for constructing a stationary coupling of a process on  $\mathbb{Z}$  and a process on  $\mathbb{N}$  and the proof in  $\S$  5.2 applies to both. Let us mention for now that this is the only place in the proof where the Toom process is required to be regular.

One final lemma is needed before we start with the proof of Theorem 2.6, this time unrelated to any coupling.

**Lemma 3.7.** Let  $\sigma$  be a Toom process on  $\mathbb{Z}$  (not necessary stationary) satisfying Condition A. Then with probability 1, if at some time  $t_0$ , the limit

$$\lim_{N \to \infty} \frac{1}{2N+1} |\{x \in [-N, N] : \sigma_{t_0}(x) = 1\}|$$

exists (" $\sigma_0$  has a density"), then for any  $t > t_0$  we also have that  $\sigma_t$  has a density, and the densities are equal.

*Proof.* Fix T and  $t_0 < T$  and examine the quantity

$$D_N := ||\{x \in [-N, N] : \sigma_T(x) = 1\}| - |\{x \in [-N, N] : \sigma_{t_0}(x) = 1\}||.$$

Then  $D_N$  is the difference between the flows of particles with sign 1 into and out of [-N, N], and since the rates of these flows are bounded by the lengths of the left stretches at -N and N we have

$$\mathbb{E}D_N \le \mathbb{E}\int_{t_0}^T l_{-N}(\sigma_s) + l_N(\sigma_s) \,\mathrm{d}s \le C$$

(where C depends on the Toom process and on T, but not on N and where the fact that the time derivative is bounded by l is again by the annealed generator, see (4)). This shows that  $\frac{1}{2N+1}D_N$  converges in  $L^1$  to 0 for all  $t_0 < T$ . To pass to almost

This shows that  $\frac{1}{2N+1}D_N$  converges in  $L^1$  to 0 for all  $t_0 < T$ . To pass to almost everywhere convergence we note that if we restrict, say, to squares  $N = M^2$ , then the Markov inequality and the Borel-Cantelli lemma together give us that

$$\frac{1}{2M^2 + 1} D_{M^2} \to 0 \quad \text{almost surely } \forall t < T.$$

But if  $\frac{1}{2N+1}|\{x\in[-N,N]:\sigma_t(x)=1\}|$  converges as  $N\to\infty$  on the squares, then by monotonicity it also converges with no restriction on the N. Finally, we note that since T was arbitrary the claim just proved also holds for all integer T simultaneously, proving that  $\frac{1}{2N+1}D_N\to 0$  for all  $t\in[0,\infty)$ .

We have gathered all necessary ingredients, and we may now start the

Proof of Theorem 2.6. Let us fix a stationary measure  $\mu$  for the Toom interface as in the statement of Theorem 2.6. The idea is to construct a coupling  $\mathbb{P}$  of  $(\sigma^1, (\sigma(p))_{p \in [0,1]})$  such that:

- (a) For  $p \in [0, 1]$ , the distribution of  $\sigma(p)$  is Ber<sub>p</sub>.
- (b) The distribution of  $\sigma^1$  is  $\mu$ .
- (c) For any p:  $\mathbf{P}(\sigma^1 \leq \sigma(p) \text{ or } \sigma^1 \geq \sigma(p)) = 1$ .
- (d) If p' > p, then  $\sigma(p') \ge \sigma(p)$  a.s.

If we then define the random variable

$$\Theta := \sup\{ p \in \mathbb{Q} : \sigma^1 \ge \sigma(p) \} = \sup\{ p : \sigma^1 \ge \sigma(p) \}, \tag{11}$$

it is tempting to believe that  $\sigma^1 = \sigma(\Theta)$ , and that the distribution of  $\sigma(\Theta)$  is a mixture of product Bernoulli's with mixing measure given by the distribution of  $\Theta$ . To turn this into a rigorous proof is a bit delicate. What follows is one possible approach.

Step 1: Let  $\mathscr{P} \subset (0,1)$  be a finite set. Let  $\nu_0 = \nu_0^{\mathscr{P}}$  be a measure on  $\{\pm 1\}^{\mathbb{Z}} \times (\prod_{p \in \mathscr{P}} \{\pm 1\}^{\mathbb{Z}})$  whose first coordinate is distributed like  $\mu$  (this is property (b)) and is independent of the others, while its other  $|\mathscr{P}|$  coordinates are distributed like  $\{\text{Ber}_p : p \in \mathscr{P}\}$  (property (a)) and are coupled to satisfy property (d). Also, let  $\nu_t$  denote the distribution on  $\{\pm 1\}^{\mathbb{Z}} \times (\prod_{p \in \mathscr{P}} \{\pm 1\}^{\mathbb{Z}})$  obtained by evolving the coupling to time t when started from the initial distribution  $\nu_0$ .

We now examine a weak\*-limit point  $\nu_{\infty} = \nu_{\infty}^{\mathscr{P}}$  of the collection of time averaged measures  $(1/T) \int_0^T \nu_t \, dt$ . The stationarity of Ber<sub>p</sub> ensures  $\nu_{\infty}$  satisfies (a), the stationarity of  $\mu$  ensures (b), and of course (d) is also preserved (say because monotonicity is equivalent to having only discrepancies of the same type, and this is preserved by the process and by taking a weak limit). By Lemma 3.6,  $\nu_{\infty}$  is a stationary coupling, so we may apply Lemma 3.1 and get property (c). We note that  $\sigma(p)$  is Bernoulli so  $\mathbb{E}[r_x(\sigma(p))^k] < \infty$  for all k and x, and ditto for  $l_x$ . By Hölder's inequality we may thus get from  $\mathbb{E}[l(\sigma^1)^{1+\epsilon}] < \infty$  (the assumption of the theorem) that  $\mathbb{E}[(\max l_y \min r_y)^{1+\epsilon}] < \infty$  (the requirement of Lemma 3.1).

To pass from a finite set to an infinite set we only need to make sure that, when  $\mathscr{P} \subset \mathscr{Q}$  then  $\nu_{\infty}^{\mathscr{Q}}$  is a limit of a subsequence of the sequence that was used to define  $\nu_{\infty}^{\mathscr{P}}$ . This ensures consistency and allows to apply the Kolmogorov extension theorem. We may thus get a measure coupling  $\sigma^1$  to  $\sigma(p)$  for a dense set of p in (0,1), say to all of the rationals, satisfying (a)-(d). Abusing notations, below when we write  $p \in \mathbb{Q}$  we implicitly mean  $p \in \mathbb{Q} \cap (0,1)$ .

**Step 2:** Define the random variable

$$\Theta := \sup\{p \in \mathbb{Q} : \sigma^1 \ge \sigma(p)\} = \inf\{p : \sigma^1 \le \sigma(p)\}$$

The equality follows from monotonicity, and also gives that  $\sigma^1$  has a density i.e. the limit

$$\lim_{N \to \infty} \frac{1}{2N+1} |\{x \in [-N, N] : \sigma^{1}(x) = 1\}|$$

exists, and is equal to  $\Theta$  since with probability one for all  $p \in \mathbb{Q}$  the density of  $\sigma(p)$  is p. Note that the existence of a density is a property of  $\sigma^1$  irrespective of any coupling of it with anything else.

Step 3: We now wish to claim that the density of  $\sigma^1$  is independent of  $\{\sigma(p): p \in \mathbb{Q}\}$ . For this we need to go through step 1 again. Step 1 starts with the measure  $\nu_0$  under which  $\sigma^1$  is independent of  $\{\sigma(p): p \in \mathcal{P}\}$ . In particular the density of  $\sigma^1$  is independent of the latter collection of random variables. By Lemma 3.7 the density is preserved during the time evolution, so  $\Theta$  exists a.s. under the measures  $(1/T) \int_0^T \nu_t dt$  and is independent of  $\{\sigma(p): p \in \mathcal{P}\}$ .

We need to be careful passing to the limit in T as, in general, weak\*-limits do not preserve the existence of a density. In this case, however, it is only the coupling that changes while the marginal distributions of each coordinate stays fixed. Because  $\Theta$  can be defined purely in terms of  $\sigma^1$ , for any  $\epsilon > 0$  and we can find some local variable E, independent of T, which approximates the density of  $\sigma^1$  up to  $\epsilon$  i.e.  $\mathbb{P}(|E - \Theta| > \epsilon) < \epsilon$ . We get that E is  $\epsilon$ -independent of  $\{\sigma(p) : p \in \mathscr{P}\}$  and this property is preserved in the limit. Thus under  $\nu_{\infty}^{\mathscr{P}}$  the density is  $2\epsilon$ -independent of  $\{\sigma(p) : p \in \mathscr{P}\}$ . As  $\epsilon$  was arbitrary we get the required property for  $\mathscr{P}$ . This implies directly the property with  $\mathscr{P}$  replaced by  $\mathbb{Q}$ .

Step 4: We now show that  $\sigma^1 = \sup\{\sigma(p) : p < \Theta\}$  under  $\nu_{\infty}$ . To see this, observe that since  $\Theta$  is independent of  $\{\sigma(p) : p \in [0,1]\}$ , we can fix it in advance. On the other hand, for any monotone coupling of Bernoulli processes and any fixed x, the random variable  $X := \sup\{p : \sigma(p)(x) = 0\}$  is uniformly distributed. Hence for  $\Theta$  fixed, the probability that  $\sup\{\sigma(p) : p < \Theta\} \neq \inf\{\sigma(p) : p > \Theta\}$  is zero since none of the individual coordinates of the monotone coupling can flip at  $\Theta$ . Further, for any value of  $\Theta$  fixed in advance,  $\bar{\sigma} := \sup\{\sigma(p) : p < \Theta\}$  is a Bernoulli- $\Theta$  process Since  $\sigma^1$  is sandwiched between  $\sup\{\sigma(p) : p < \Theta\}$  and  $\inf\{\sigma(p) : p > \Theta\}$ , the foregoing discussion implies get that  $\sigma^1 = \sigma(\Theta)$  and  $\Theta$  is independent of  $\{\sigma(p)\}$ . In other words,  $\sigma^1$  is distributed as a mixture of Bernoulli processes. This proves the theorem.

Remark 3.8. Surprisingly, perhaps, the standard coupling of Bernoulli processes (i.e. the coupling where each site is coupled independently of the others) is not stationary to the Toom process (For example, take a finite system with periodic boundary condition. Then it is straightforward to write a formula for the number of incoming arrows for any given configuration and see that it is not constant). In other words, the process  $\sigma(p)$  which we

analyzed during the proof is a collection of Bernoulli processes, coupled by a non-standard monotone coupling.

## 4 Proof of Theorem 1.2

Recall that we wish to study the process on  $\mathbb{N}$  away from 0. For this we consider the coupling process defined on the configuration space  $\{\pm 1\}^{\mathbb{N}} \times \{\pm 1\}^{\mathbb{Z}}$ . We will write  $\sigma = (\sigma^1(x), \sigma^2(x))$  for the configuration  $\sigma \in \{\pm 1\}^{\mathbb{N}} \times \{\pm 1\}^{\mathbb{Z}}$ . We use the notation  $\mathbf{D}$ ,  $\mathbf{D}^+$  and  $\mathbf{D}^-$  for discrepancies set out at the beginning of § 3 with the understanding that *all* vertices  $x \leq 0$  host discrepancies at all times.

Throughout this section, we let p be the unique solution in (0,1) to the equation

$$\left(\frac{1-p}{p}\right)^2 = \frac{\lambda_+}{\lambda_-} \quad .$$
(12)

Let  $\nu$  be a probability measure on  $\{\pm 1\}^{\mathbb{N}} \times \{\pm 1\}^{\mathbb{Z}}$  stationary for the coupling process and with respective marginals  $\pi_{\infty}$ , Ber<sub>p</sub>. We know such measures exist by Lemma 3.6, see proof in § 5.2. (There is in fact a unique such measure, though we do not need to use this explicitly). From the next proposition, Theorem 1.2 follows easily.

**Proposition 4.1.** With  $\nu$  as above,

$$\lim_{x \to \infty} \nu(x \in \mathbf{D}) = 0.$$

Before discussing Proposition 4.1 further, let use it to attend to Theorem 1.2.

*Proof of Theorem 1.2.* Consider a local function f and recall the shifts  $\tau_x$ . By the definition of  $\mathbf{D}$ ;

$$\left| \mathbb{E}_{\pi_{\infty}}[f \circ \tau_x] - \mathbb{E}_{\mathrm{Ber}_p}[f \circ \tau_x] \right| \le \|f\|_{\infty} \sum_{y \in x + \mathrm{Supp} f} \nu(y \in \mathbf{D}).$$

By Proposition 4.1 the RHS tends to 0 as x tends to infinity. But since  $\operatorname{Ber}_p$  is invariant under spatial shifts, this implies the push forward of  $\pi_{\infty}$  by  $\tau_x$  converges weakly to  $\operatorname{Ber}_p$  as x tends to infinity.

Proof of Proposition 4.1. As in the proof of Lemma 3.4, the crucial quantity that we will analyze is the discrepancy flow through a point x, which we will again denote by  $j_x$ . Recall that  $j_x$  is the averaged infinitesimal rate at which a discrepancy in  $(-\infty, x]$  jumps to  $(x, \infty)$ . Let us note a few simple properties of j. First,  $\nu(x \in \mathbf{D}) \leq j_x$  because whenever there is a discrepancy at x it jumps to  $(x, \infty)$  with rate at least the rate of the Poisson arrivals at x. Next, note that  $j_x$  is always finite: both  $\sigma^1$  and  $\sigma^2$  satisfy the local

Condition A,  $\sigma^1$  because  $l_x(\sigma^1) \leq x$  and in particular is finite, while  $\sigma^2$  is a Bernoulli process. By Lemma 3.3, this implies that  $j_x$  is finite.

Next, again as in the proof of Lemma 3.4, let  $a_x$  be the rate of annihilations at x, i.e. the averaged infinitesimal rate at which a + discrepancy at some y < x moves on a - discrepancy at x causing both to disappear, and vice versa. Recall (7) i.e.  $j_{x-1} - j_x = a_x$ . Let k be some fixed parameter and let  $x \in \mathbb{N}$ . One of the following three events must occur:

- 1. The interval  $I_x := [\max(0, x k^2), x]$  contains two discrepancies with opposing signs.
- 2. The last k discrepancies before x are of the same sign (and case 1 did not happen).
- 3. The last k discrepancies before x are not of the same sign (and case 1 did not happen)

We divide the set of discrepancies **D** and the infinitesimal rates of discrepancy flow  $j_x$  into three parts accordingly,  $\mathbf{D} = \mathbf{D}^1 \cup \mathbf{D}^2 \cup \mathbf{D}^2$  and  $j_x = j_x^1 + j_x^2 + j_x^3$ . Thus, for example,

$$\mathbf{D}^1 = \{ x \in \mathbf{D} : I_x \cap \mathbf{D}^+ \neq \emptyset, \ I_x \cap \mathbf{D}^- \neq \emptyset \}$$

and similarly for the other  $\mathbf{D}^i$ . Let us also give an example with j:  $j_x^1$  is the averaged infinitesimal rate that discrepancies jump from  $(-\infty, x]$  to  $(x, \infty)$  while there are two discrepancies of opposite signs in  $I_x$  (we are not bothered about whether the discrepancy that performed the jump is in fact one of those).

Let us isolate the estimates used to control D, j in two lemmas. We note first of all that

$$\nu(x \in \mathbf{D}^i) \le j_x^i \qquad i = 1, 2, 3.$$

following the same argument we used above to show that  $\nu(x \in \mathbf{D}) \leq j_x$ . This means that our control on  $\mathbf{D}^i$  in these lemmas follows from our control on  $j^i$ .

**Lemma 4.2.** For every k there exists a  $C_k$  such that  $\mathbb{E}|\mathbf{D}^1| \leq C_k$ . Further,

$$\sum_{x=1}^{\infty} j_x^1 \le C_k$$

The proof (see below) proceeds by observing that a configuration contributing to  $j_x^1$  has a positive probability to lead to an annihilation. This part of the argument would work equally well with any value of p. It is in the next lemma that the specific choice (12) becomes crucial.

**Lemma 4.3.** There exist numbers  $\epsilon_k \downarrow 0$  such that  $\nu(|\mathbf{D}^2 \cap [1, n]|) \leq \epsilon_k n$ . Further,  $j_x^2 \leq \epsilon_k$  for all x > k.

We defer the proof of both lemmas and instead first show how they imply Proposition 4.1. We start with a preliminary claim

**Lemma 4.4.** 
$$\lim_{n\to\infty} \frac{1}{n} \nu(|\mathbf{D} \cap [1,n]|) = 0$$

*Proof.* Fix  $k \in \mathbb{N}$  and decompose **D** into the 3 parts using this k. By the "**D** parts" of Lemmas 4.2 and 4.3 we see that

$$\nu(|(\mathbf{D}^1 \cup \mathbf{D}^2) \cap [1, n]|) \le C_k + n\epsilon_k.$$

As for  $\mathbf{D}^3$ , the conditions of case 3 imply that the interval  $[\max(0, x - k^2), x]$  contains less than k discrepancies. It follows from this that  $|\mathbf{D}^3 \cap [1, n]| \leq n/k + k^2$ . This implies

$$\limsup_{n} (1/n)\nu(|\mathbf{D} \cap [1,n]|) \le \epsilon_k + 1/k.$$

Taking  $k \to \infty$ , the lemma is proved.

To move from the averaged result of Lemma 4.4 to the non-averaged result we argue as follows. Let  $\epsilon > 0$ . First fix some k sufficiently large such that the  $\epsilon_k$  from Lemma 4.3 satisfies  $\epsilon_k < \epsilon$ . By Lemma 4.2 we can choose  $x_0$  such that for all  $x > x_0$  one has that  $j_x^1 < \epsilon$ . Next, let M be some parameter (to be fixed shortly and depending only on k and on  $\lambda^+/\lambda^-$ ). By Lemma 4.4 we can further find some  $x > x_0$  such that

$$\sum_{y=x-M}^{x} \nu(y \in \mathbf{D}) < \epsilon/k \tag{13}$$

(we assume here that x > M and, while we are at it, also  $x > k^2$ ).

We will now show that  $j_x < C\epsilon$ . Since  $j_x^1$  and  $j_x^2$  are already given to us as satisfying such a bound we need only estimate  $j_x^3$ . For this purpose we note that  $\sigma^2$  is a Bernoulli process, so there are constants c, C > 0, such that one cannot create an interval of consecutive equal signs larger than Ck before x by changing the sample  $\sigma^2$  at less than k deterministic locations, except with probability  $> 1 - e^{-ck}$ . Fix the parameter M from the last paragraph to be this Ck and denote this event by  $G_k$ . Split  $j_x^3 = j_x^{3.1} + j_x^{3.2}$ , where  $j_x^{3.1}$  denotes the rate at which discrepancies pass from  $(-\infty, x]$  to  $(x, \infty)$  by arrivals in [x - M, x] and  $j_x^{3.2}$  is the remainder. It follows that

$$j_x^{3.1} \le (M+1) \sum_{y=x-M}^x \nu(y \in \mathbf{D}).$$
 (14)

because contributions to  $j_x^{3.1}$  require at least one discrepancy in [x-M,x] and also require a Poisson arrival in this interval. The factor M+1 in the estimate comes as an upper bound on the latter rate. Using (13) and the fact that M=Ck we get  $j_x^{3.1} \leq C\epsilon$ .

To bound  $j_x^{3.2}$  notice that a pair of particles of opposing signs in both configurations acts as a barrier for discrepancies. Hence if an arrival occured at some y and pushed a discrepancy beyond x, then either  $\sigma_1$  or  $\sigma_2$  must have constant sign between x and y. Since we are in case 3 there must be less than k discrepancies between x and y (two discrepancies of opposing signs are also a barrier), so we have that  $\sigma_2$  must have less than k particles of some kind. This allows to bound  $j_x^{3.2}$  by examining  $\sigma_2$  only, and  $\sigma_2$  is a Bernoulli process. The probability that  $\sigma_2$  restricted to [y,x] has less than k particles of some kind can be bounded above roughly by  $C \exp(-ck - c(x-y)/k)$  — recall the definition of M — and we get

$$j_x^{3.2} \le C \sum_{y=-\infty}^{x-M-1} e^{-ck-c(x-y)/k} \le Ce^{-ck}.$$

Taking everything together we get

$$j_x \le C\epsilon + Ce^{-ck}$$
.

and since we assumed k is sufficiently large (depending on  $\epsilon$ ) we may incorporate the  $Ce^{-ck}$  into the other term and conclude, as promised, that  $j_x \leq C\epsilon$ .

Proposition 4.1 is now proved because  $j_x$  are decreasing, by (7). We get that for all y > x  $j_y \le C\epsilon$  and since  $\epsilon$  was arbitrary  $j_y \to 0$ . As already remarked  $\nu(x \in \mathbf{D}) \le j_x$  so we also get  $\nu(x \in \mathbf{D}) \to 0$ .

Proof of Lemma 4.2. An important corollary of (7) is that  $j_x$  is decreasing and that  $\sum a_x < \infty$ . The proof of the lemma then follows by comparing  $j_x^1$  and  $a_x$ .  $j_x^1$  is the rate at which discrepancies flow through x while there are two discrepancies with opposing signs in the interval  $I_x = [\max(0, x - k^2), x]$ . This couple of discrepancies acts as a barrier, so any arrival that pushed a discrepancy beyond x must be in  $I_x$ . Hence

$$j_x^1 \leq |I_x| \cdot \nu$$
 (there are two discrepancies of opposing signs in  $I_x$ ).

In the language of Lemma 3.5, this event implies that for some  $y, z \in I_x$ , the event  $E_{y,z}$  occurred. Applying Lemma 3.5 gives

$$j_x^1 \le k^2 \sum_{y,z \in I_x} \nu(E_{y,z}) \le \frac{k^2}{c(k)} \sum_{y,z \in I_x} \sum_{w=y}^z a_w \le C(k) \sum_{w \in I_x} a_w$$

as required.

## 4.1 Proof of Lemma 4.3

As already mentioned, it is in this part of the proof that the relation between p and  $\lambda$  is used. We start by indicating the reason for this relation. First denote by  $H_x^{\pm}(t)$  the total number of  $\pm$  discrepancies which jumped from  $(-\infty, x]$  to  $(x, \infty)$  between time 0 and time t. Let  $H_x(t) = H_x^+(t) + H_x^-(t)$  be the total flow and let  $K_x(t) = H_x^+(t) - H_x^-(t)$  be the signed flow through x. We will only need the signed flow at 0 (where it is a function of the Bernoulli process alone) and so abbreviate  $K(t) = K_0(t)$ .

**Lemma 4.5.** If 
$$((1-p)/p)^2 = \lambda_+/\lambda_-$$
 then  $\mathbb{E}[K(t)] = 0$ .

Proof. This is a straightforward calculation: the infinitesimal rate at which + particle enter  $\mathbb{N}$  is  $\lambda_+ l_1(\sigma^2) \mathbb{1}\{\sigma^2(0) = 1\}$  (recall that  $l_1$  is the length of the block of identical spins from 0 to its left). Since  $\sigma^2$  is a Bernoulli-p process,  $\mathbb{E}[l_1(\sigma^2) \cdot \mathbb{1}\{(\sigma^2(0) = 1\}] = p^2$ . The same holds for the – particles and we get  $\mathbb{E}(K(t)) = p^2\lambda_+ - (1-p)^2\lambda_-$ .

We next need two preliminary results which are interesting in their own right. It is possible to show that K satisfies a functional CLT under proper rescaling, see [4], but for our current purposes, the following diffusive bound suffices. Its proof is supplied in § 5.1.

**Lemma 4.6.** There exists C > 0 such that for all  $t \in \mathbb{R}$ ,

$$\mathbb{E}_{\mathrm{Ber}_n}[(K(t) - K(0))^2] \le Ct.$$

We also need a bound on the total flux of discrepancies across a fixed vertex for the process on  $\{\pm 1\}^{\mathbb{Z}}$ . Set  $\mathcal{N}([a,b]) = H_0(b) - H_0(a)$ .

**Lemma 4.7.** For sufficiently small  $\gamma$ ,

$$\mathbb{E}_{\mathrm{Ber}_p}[\exp(\gamma \mathcal{N}(I)/|I|)] \leq C, \quad \text{for any } I.$$

In particular

$$\mathbb{P}_{\mathrm{Ber}_p}(\mathcal{N}(I) \ge N) \le C(m)(|I|/N)^m \tag{15}$$

The proof is also deferred to  $\S 5.1$ .

To explain the relation between  $j_x^2$  and K we need the following definition. For a partition  $\pi$  of [0,t] into intervals, let

$$\mathscr{K}(\pi) = \sum_{I \in \pi} |\Delta K(I)| \mathbb{1}\{|\Delta K(I)| \ge k\}$$

where  $\Delta K(I) := K(\sup I) - K(\inf I)$  and let

$$\mathscr{K}^*(t) = \mathscr{K}^*(k,t) = \sup_{\pi} \mathscr{K}(\pi).$$

With these definitions we can now claim

## **Lemma 4.8.** For all $x \ge 1$ and t, $H_x^2(t) \le x + \mathcal{K}^*(t)$ .

Here  $H_x^2(t)$  is the number of discrepancies of "type 2" which passed through x until time t, using the same classification of discrepancies into 3 types we used on page 27 to define  $j_x^2$ .

*Proof.* The x term in the lemma is a crude bound for "original" discrepancies i.e. for discrepancies which existed at time 0 in [1, x]. We ignore these discrepancies and order the others by their time of crossing x.

In this proof it will be convenient to think about discrepancies as having a fixed order. Examine a discrepancy. When it moves (to the right only!), it can collide with the first discrepancy to its right. If the second discrepancy has the opposite sign, the two annihilate leaving behind two vertices each having spin agreement. If the two discrepancies have the same sign we shall use the interpretation that the first one takes the place of the second one, and the second starts moving. This can create a chain reaction, but we note that as soon as there is an annihilation, the process stops. In any case, the order of discrepancies never changes.

Let us now use this interpretation to examine a stretch of  $\ell$  type 2 discrepancies with the same sign contributing to  $j_x^2$ , say with sign + for concreteness. The clock ring which precipitates the jump contributing to  $j_x^2$  does not have to occur in  $I_x$  and could come from a particle of either sign. However, the chain reaction which occurs can only involve the discrepancies with sign +. This follows from our definition; just before the jump occurred, the last k discrepancies to the left of x were of the same sign. Since discrepancies of different types do not propagate motion when they collide, we conclude that the discrepancies of  $I_x$  must be + discrepancies.

Hence a stretch of  $\ell$  discrepancies of type 2 which crossed x actually corresponds to  $\ell + k - 1$  discrepancies of sign +, the  $\ell$  discrepancies which crossed as well as another (at least) k - 1 which remain in  $I_x$  at the time of the last crossing. Let a be the time when the first one entered the system and b the time when the last one did.

We claim that  $K(b) - K(a) = \ell + k - 1$ . This relies on the fact that signed discrepancy sum is preserved by our dynamics: discrepancies survive until they are annihilated in pairs of + and -. So if we see  $\ell + k - 1$  consecutive + discrepancies at some space-time point (x,t), we must have started with a signed sum of  $\ell + k - 1$ . (the term "consecutive" might be slightly misleading here, since some of them are consecutive in the time of crossing of x and others are consecutive in space at a given time, but for the claim on K(b) - K(a) this does not matter).

Finally, different stretches of discrepancies of type 2 crossing x must correspond to different time intervals [a, b]. Indeed, examine the last + discrepancy in one stretch and the first - discrepancy in the next. Being of different sign they cannot cross one another without annihilating, and since we know the + arrived at  $I_x$  before the -, the - must have started after it.

Thus the stretches of discrepancies of type 2 crossing x form disjoint collection of subintervals of [0,t], each of which has  $|\Delta K(I)| \geq k$ . Completing this collection to a partition  $\pi$ , the lemma is proved.

Lemma 4.3 will thus be proved once we show

#### Lemma 4.9.

$$\limsup_{k \to \infty} \limsup_{t \to \infty} \frac{1}{t} \mathbb{E}_{\mathrm{Ber}_p} [\mathscr{K}^*(k,t)] = 0.$$

Note that we have written  $\mathbb{E}_{\mathrm{Ber}_p}$  instead of  $\mathbb{E}$  since K depends only on  $\sigma^2$ .

*Proof.* The proof of this lemma relies on a separation of scales. For each partition  $\pi$  of [0,t], let us split the sum  $\mathcal{K}(\pi)$  according to interval sizes:

$$\mathcal{K}(\pi) = \underbrace{\sum_{I \in \pi: |I| \le k^{1/2}} |\Delta K(I)| \mathbb{1}\{|\Delta K(I)| \ge k\}}_{\mathcal{K}^{1}(\pi)} + \underbrace{\sum_{I \in \pi: |I| > k^{1/2}} |\Delta K(I)| \mathbb{1}\{|\Delta K(I)| \ge k\}}_{\mathcal{K}^{2}(\pi)}$$
(16)

We will use separate mechanisms to bound each of  $\mathcal{K}^1$  and  $\mathcal{K}^2$  uniformly in  $\pi$ .

Let us first attend to  $\mathcal{K}^1(\pi)$ . The idea here is simple: since a contributing interval I is small relative to k, it necessitates too many Poisson arrivals, at least k, in I. This is a rare event, and gets exponentially rarer as  $|I| \to 0$ . This fact allows us to handle all partitions simultaneously via a properly chosen infinite covering of [0, t].

For each  $j \in \mathbb{Z}$  let

$$X_j = \sum_{i=0}^{\lfloor t2^j \rfloor} \mathcal{N}([i2^j, (i+1)2^j]) \mathbb{1} \{ \mathcal{N}([i2^j, (i+1)2^j]) \ge k/2 \}.$$

Then for any  $\pi$ , exploiting that  $\mathcal{N}(I)$  is the total variation of K(I),

$$\sup_{\pi} \mathcal{K}^{1}(\pi) \le 2 \sum_{j < \log_{2}(k^{1/2}) + 1} X_{j}.$$

Taking expected values on both sides and using stationarity

$$\frac{1}{t} \mathbb{E}_{\mathrm{Ber}_p} \left[ \sup_{\pi} \mathcal{K}^1(\pi) \right] \le \sum_{j \le \log_2(k^{1/2}) + 1} 2^{1-j} \mathbb{E}_{\mathrm{Ber}_p} \left[ \mathcal{N}([0, 2^j]) \mathbb{1} \{ \mathcal{N}([0, 2^j]) \ge k \} \right]. \tag{17}$$

Using (15), we find that for k sufficiently large, the RHS of (17) is summable and that, moreover, it tends to 0 as k tends to  $\infty$ .

To bound  $\mathcal{K}^2(\pi)$  let us introduce a reference partition  $\rho = \{[ik^{1/4}, (i+1)k^{1/4}]\}$  of [0,t] (shorten the last interval if necessary). Let  $\rho' \subset \rho$  be the collection of intervals which

contain an endpoint of some interval from  $\pi$ . Then we can write

$$\mathcal{K}^2(\pi) \leq \underbrace{\sum_{I \in \rho} |\Delta K(I)|}_{\mathbf{I}} + 2\underbrace{\sum_{I \in \rho} \mathcal{N}(I)\mathbb{1}\{\mathcal{N}(I) \geq k^{1/3}\}}_{\mathbf{II}} + 2\underbrace{\sum_{I \in \rho'} \mathcal{N}(I)\mathbb{1}\{\mathcal{N}(I) < k^{1/3}\}}_{\mathbf{III}}.$$

Term III is bounded by  $4tk^{-1/2}k^{1/3}$  since the total number of intervals  $I \in \pi$  with  $|I| \ge k^{1/2}$  is at most  $2tk^{-1/2}$ . For the other two terms, we take the expectation and use the fact that they no longer depend on  $\pi$ . For II, the bound (15), yields

$$\mathbb{E}_{\text{Ber}_{p}}[\Pi] \le 2tk^{-1/4} \times C(m)(k^{1/4}/k^{1/3})^{m}$$

which vanishes as  $k \to \infty$  by choosing m sufficiently large. For the first term, we argue

$$\mathbb{E}_{\mathrm{Ber}_p}[I] \le tk^{-1/4} (\mathbb{E}_{\mathrm{Ber}_p}[|K(k^{1/4}) - K(0)|^2])^{1/2} \le Ctk^{-1/8}$$
(18)

where the first inequality is by stationarity and Cauchy-Schwarz, and the second follows from the diffusive moment estimate of Lemma 4.6. Hence we have obtained

$$\lim_{k \to \infty} \sup_{\pi} (1/t) \sup_{\pi} \mathcal{K}^2(\pi) = 0,$$

Combining with the analogous bound on  $\mathcal{K}^1$ , the assertion of the lemma follows.

# 5 Auxiliary results

### 5.1 Proofs of lemmas 4.6 and 4.7.

Proof of Lemma 4.6. Let f be the infinitesimal drift of K i.e.  $f(t) = \sigma(0)\lambda_{\sigma(0)}l_1(\sigma)$  (in other words, if  $\sigma(0) = 1$  then  $f = \lambda_+ l_1$  and otherwise  $f = -\lambda_- l_1$ ). Using the Cauchy-Schwarz inequality,

$$\mathbb{E}\left[K(t)^{2}\right] \leq 2\mathbb{E}\left[\left(K - \int_{0}^{t} f(\sigma_{s}) \,ds\right)^{2}\right] + 2\mathbb{E}\left[\left(\int_{0}^{t} f(\sigma_{s}) \,ds\right)^{2}\right].$$

(recall that K and hence f are functions of the Bernoulli process, so all expectations are likewise with respect to it). The first term is a martingale, so its expected square is its quadratic variation. By direct calculation this is  $\int_0^t \lambda_{\sigma(0)} l_1(\sigma)$ , which is bounded by Ct. For the second term, we use stationarity to bound it by

$$2t \int_0^\infty |\mathbb{E}[f(\sigma_s)f(\sigma_0)]| \, \mathrm{d}s.$$

We shall show  $\mathbb{E}_{\mathrm{Ber}_p}[f(\sigma_s)f(\sigma_0)]$  decays exponentially in s to complete the proof.

We want to apply Theorem 2.10. One slight complication is that the function f has unbounded support. To handle this issue, let  $f_n$  be the approximation to f above by replacing  $l_1$  with  $\min\{l_1, n\}$ , which is a local function. Then, by inspection,

$$\mathbb{E}[(f - f_n)^2] \le Ce^{-cn}$$

so that, for any n, Cauchy-Schwarz yields

$$|\mathbb{E}[f(\sigma_s)f(\sigma_0)] - \mathbb{E}[f_n(\sigma_s)f_n(\sigma_0)]| \le Ce^{-cn/2}.$$

The autocorrelation of  $f_n$  is handled by Theorem 2.10, with r = n, so that

$$|\mathbb{E}[f(\sigma_t)f(\sigma_0)]| \le C(e^{-cn/2} + e^{n-ct})$$

and, choosing  $n = \frac{1}{2}ct$ , we get

$$|\mathbb{E}[f(\sigma_t)f(\sigma_0)]| \leq Ce^{-ct}$$
.

Lemma 4.6 follows.  $\Box$ 

Proof of Lemma 4.7. Recall that we wish to estimate  $H = H_0(t)$ , the number of particles entering  $\mathbb{N}$  in [0,t]. We define  $H^L(t)$  to be the number of particles which enter from (-L,0), and note that  $H^L \nearrow H$  so by monotone convergence  $\mathbb{E}(\exp(\gamma H^L)) \to \mathbb{E}(\exp(\gamma H))$ .

Next denote the drift of  $H^L$  by  $v = v^L(t)$  i.e.

$$v_t^L := \int_0^t \min(l_1(\sigma_s), L) \, \mathrm{d}s.$$

We first find an estimate on  $\mathbb{E}(\exp(\alpha v))$ . We Taylor expand the exponential

$$\mathbb{E}(e^{\alpha v}) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \mathbb{E}(v^k),$$

integrate each term and use Hölder's inequality and stationarity to get

$$\mathbb{E}(v^k) \le \int_0^t \cdots \int_0^t \mathbb{E}(l_1(\sigma_{s_1}) \cdots l_1(\sigma_{s_k})) \, \mathrm{d}s_1 \cdots \mathrm{d}s_k \le t^k \mathbb{E}(l_1(\sigma_0)^k)$$

so

$$\mathbb{E}(e^{\alpha v}) \le \mathbb{E}_{Ber_v}[e^{\alpha t l_1}]. \tag{19}$$

which is finite for  $\alpha < c(p)/t$ .

Next, one verifies by direct computation (omitted here) that, for any  $\beta > 0$ ,

$$Z_{\beta}^{L}(t) := \exp\left\{\beta H^{L}(t) - (e^{\beta} - 1)v_{t}^{L}\right\}$$

is a martingale. To exploit this, fix  $\beta \in \mathbb{R}$  and set  $\alpha = (1/2)(e^{2\beta} - 1)$  and write

$$\mathbb{E}_{\mathrm{Ber}_p}[\exp(\beta H^L)] = \mathbb{E}_{\mathrm{Ber}_p}\left[\exp(\beta H^L - \alpha v)\exp(\alpha v)\right].$$

Applying the Cauchy-Schwarz inequality and noting that  $2\alpha = e^{2\beta} - 1$ 

$$\mathbb{E}_{\mathrm{Ber}_p}[\exp(\beta H^L(t))] \le \mathbb{E}_{\mathrm{Ber}_p}[Z_{2\beta}(t)]^{1/2} \mathbb{E}_{\mathrm{Ber}_p}[\exp(2\alpha v)]^{1/2}.$$

Since the first term on the RHS evaluates to 1 (being the expectation of a martingale), the lemma follows from (19) and  $\mathbb{E}(\exp(\alpha H^L)) \to \mathbb{E}(\exp(\alpha H))$  upon choosing  $\beta = \gamma/t$  and noting that if  $\gamma$  is sufficiently small then  $2\alpha t$  is small enough to ensure that the right hand side of (19) is finite.

## 5.2 Existence of a stationary coupling

Our goal in this section is to prove Lemma 3.6. Recall that it stated that if  $(\mu^1, F^1)$  and  $(\mu^2, F^2)$  are two stationary, regular Toom processes and if  $\nu$  is any coupling of them, then any subsequential weak\*-limit of  $\frac{1}{T} \int_0^T \nu_t$  is a stationary coupling of  $(\mu^1, F^1)$  and  $(\mu^2, F^2)$ . Here  $\nu_t$  is the result of applying the coupling to  $\nu$  for time t. As this is the only place in the paper where regularity of Toom processes is used, let us recall its definition (Definition 2.5 on page 9): a Toom process  $(\mu, F)$  is called regular if  $F_t$  can be written as the  $\mu \times \mathbb{P}$ -limit in measure of functions  $F_t^L$  such that  $F_t^L(\cdot, \omega)$  is continuous for almost all  $\omega$ .

*Proof.* As the lemma is used both to construct a coupling between two processes on  $\mathbb{Z}$  (in  $\S$  3) and between a process on  $\mathbb{Z}$  and a process on  $\mathbb{N}$  (in  $\S$  4), let us denote by  $\Xi$  the corresponding state space i.e.  $(\{\pm 1\}^{\mathbb{Z}})^2$  or  $\{\pm 1\}^{\mathbb{N}} \times \{\pm 1\}^{\mathbb{Z}}$ , as the case may be. Let  $T_n$  be the sequence over which we assume said measures converge weakly\* and denote

$$\lambda_n = \frac{1}{T_n} \int_0^{T_n} \nu_t \, \mathrm{d}t \qquad \lambda_\infty = \lim \lambda_n.$$

Fix some t. By the definition of regularity, we may write  $F_t^1$  as a limit in measure of functions  $F_t^{1,L}$  such that  $F_t^{1,L}(\cdot,\omega)$  is continuous for almost all  $\omega$ . Repeat for  $F^2$ . Let us

define the corresponding operators on probability measures on  $\Xi$  i.e.

$$P_t \lambda(E) = \int \mathbb{1}_E(F_t(\eta^1, \eta^2, \omega)) \, d\lambda(\eta^1, \eta^2) \, d\mathbb{P}(\omega) \qquad \forall E \subset \Xi \text{ Borel}$$

$$P_t^L \lambda(E) = \int \mathbb{1}_E(F_t^L(\eta^1, \eta^2, \omega)) \, d\lambda(\eta^1, \eta^2) \, d\mathbb{P}(\omega) \qquad \forall E \subset \Xi \text{ Borel}.$$

where here and below we denote

$$F_t(\eta^1, \eta^2, \omega) = (F_t^1(\eta^1, \omega), F_t^2(\eta^2, \omega))$$

and ditto for  $F^L$ . For conciseness, fix t and remove it from the notations  $P_t$  and  $P_t^L$ . We now fix some  $f:\Xi\to\mathbb{R}$  and write

$$P\lambda_{\infty}(f) - \lambda_{\infty}(f) =$$

$$(P\lambda_{\infty}(f) - P^{L}\lambda_{\infty}(f)) + (P^{L}\lambda_{\infty}(f) - P^{L}\lambda_{n}(f)) + (P^{L}\lambda_{n}(f) - P\lambda_{n}(f))$$

$$+ (P\lambda_{n}(f) - \lambda_{n}(f)) + (\lambda_{n}(f) - \lambda_{\infty}(f)) = I + II + \dots + V.$$

We will now bound the different terms. We need the following lemma.

**Lemma 5.1.** For every continuous  $f:\Xi\to\mathbb{R}$  and every  $\epsilon>0$  there exists an L such that

$$\int |f(F_t^L(\eta^1, \eta^2, \omega)) - f(F_t(\eta^1, \eta^2, \omega))| \, \mathrm{d}\lambda(\eta^1, \eta^2) \, \mathrm{d}\mathbb{P}(\omega) \le \epsilon \tag{20}$$

for any  $\lambda$  with marginals  $\mu^1$  and  $\mu^2$  and for every t > 0.

*Proof.* Let  $\delta > 0$  be some parameter to be fixed later. Let  $B^1 \subset \Xi \times \Omega$  be the set

$$B^{1} = \{ (\eta^{1}, \eta^{2}, \omega) : |F_{t}^{1,L}(\eta^{1}, \omega) - F_{t}^{1}(\eta^{1}, \omega)| > \delta \}$$

and similarly  $B^2$  with  $F^2$  instead of  $F^1$ . Since  $F^{1,L} \to F^1$  in measure, and since  $B^1$  does not depend on  $\eta^2$  or on the coupling, we get that for L sufficiently large  $\nu \times \mathbb{P}(B^1) < \delta$ , and similarly for  $B^2$ . Examine now the integral in (20) and write

$$\int = \int_{B^1} + \int_{B^2} + \int_{\Xi \times \Omega \setminus (B^1 \cup B^2)}.$$

The first and second terms are each bounded by  $\delta ||f||_{\infty}$  since the measures of  $B^i$  are small. The last term is bounded by the modulus of continuity of f i.e. by  $\max\{|f(\eta) - f(\eta')| : d(\eta, \eta') \le 2\delta\}$ . If we pick  $\delta$  sufficiently small such that the sum of the three terms is smaller than  $\epsilon$ , and the lemma is proved.

We return to bounding the terms I,..., V. To bound terms I and III we use Lemma 5.1 to get that I, III  $\leq \epsilon$  whenever L is sufficiently large (depending on f and  $\epsilon$ , but inde-

pendent of n). Term V converges to 0 as  $n \to \infty$  because  $\lambda_n \to \lambda_\infty$  weakly\* and f is continuous. For term II we write

$$II = \int \left( \int f(F_t^L(\eta^1, \eta^2, \omega)) \, d\lambda_{\infty}(\eta^1, \eta^2) - \int f(F_t^L(\eta^1, \eta^2, \omega)) \, d\lambda_n(\eta^1, \eta^2) \right) d\mathbb{P}(\omega).$$

The functions integrated are continuous (for almost every  $\omega$ ), so the inner term converges to 0 for almost every  $\omega$  as  $n \to \infty$ . It is also bounded since f is bounded. By the bounded convergence theorem we get that II  $\to$  0 as  $n \to \infty$ .

Finally, term IV is bounded by the observation that  $P_t\nu_s = \nu_{t+s}$ . Let us postpone the proof of this fact (which is just playing with definitions) and write

$$\begin{split} \mathrm{IV} &= \frac{1}{T_n} \bigg( P_t \int_0^{T_n} \nu_s \, \mathrm{d}s - \int_0^{T_n} \nu_s \, \mathrm{d}s \bigg)(f) = \frac{1}{T_n} \bigg( \int_0^{T_n} \nu_{s+t} \, \mathrm{d}s - \int_0^{T_n} \nu_s \, \mathrm{d}s \bigg)(f) \\ &= \frac{1}{T_n} \bigg( \int_{T_n}^{T_n+t} \nu_s \, \mathrm{d}s - \int_0^t \nu_s \, \mathrm{d}s \bigg)(f) \leq \frac{2t}{T_n} \max |f| \xrightarrow[n \to \infty]{} 0. \end{split}$$

Let us wrap up the calculation. We fix L sufficiently large so that  $I + III \le \epsilon$  uniformly in n. We take  $n \to \infty$  and get  $|P\lambda_{\infty}(f) - \lambda_{\infty}(f)| \le \epsilon$ . Since  $\epsilon$  was arbitrary, these are actually equal. Since f was an arbitrary continuous function,  $P_t\lambda_{\infty} = \lambda_{\infty}$ . Since t was arbitrary, this is the required stationarity.

We still need to show  $P_t \nu_s = \nu_{t+s}$ . We write

$$P_{t}\nu_{s}(f) = \int f(F_{t}(\eta^{1}, \eta^{2}, \omega)) \, d\nu_{s}(\eta^{1}, \eta^{2}) \, d\mathbb{P}(\omega)$$

$$= \int f(F_{t}(F_{s}(\eta^{1}, \eta^{2}, \omega'), \omega)) \, d\nu(\eta^{1}, \eta^{2}) \, d\mathbb{P}(\omega') \, d\mathbb{P}(\omega)$$

$$= \int f(F_{t}(F_{s}(\eta^{1}, \eta^{2}, \omega), S_{s}\omega)) \, d\nu(\eta^{1}, \eta^{2}) \, d\mathbb{P}(\omega)$$

$$= \int f(F_{t+s}(\eta^{1}, \eta^{2}, \omega)) \, d\nu(\eta^{1}, \eta^{2}) \, d\mathbb{P}(\omega) = \nu_{t+s}(f).$$

The first equality is the definition of  $P_t$ . The second is the definition of  $\nu_s$ . The third uses that  $F_s$  is  $\mathscr{F}_s$ -measurable, so we can replace the two independent Poisson processes  $\omega$  and  $\omega'$  with a single Poisson process, and use  $\omega$  for the first and  $S_s\omega$  for the second (recall that  $S_s$  are the natural time shifts of the Poisson process). The fourth equality is the semigroup property for F.

## 5.3 Proof of Lemma 2.11

We begin this section with a technical lemma. Let Q(y,t) be the number of times  $\sigma_s(y)$  changed in the time interval [0,t] and  $N_y$  the Poisson process at y.

**Lemma 5.2.** Let  $\sigma$  be a Toom process,  $y \in \mathbb{Z}$  and t > 0. Then

$$\mathbb{E}Q(y,t) \le \int_0^t \mathbb{E}l_y(\sigma_s) \, ds + \mathbb{E}N_y(t)$$

(possibly in the sense of  $\infty \leq \infty$ ).

*Proof.* Let x < y and let Q(x, y, t) be the number of Poisson arrivals in x in the time interval [0, t] that caused the value of  $\sigma(y)$  to change (recall from the definition of a Toom process that every change in y must correspond to some x < y and some Poisson arrival at x at time t such that  $\sigma_{t-}(x) = \sigma_{t-}(x+1) = \cdots = \sigma_{t-}(y-1) = -\sigma_{t-}(y)$ ). Then

$$\mathbb{E}Q(y,t) = \sum_{x < y} \mathbb{E}Q(x,y,t) + \mathbb{E}N_y(t)$$

and

$$\int_0^t \mathbb{E}l_y(\sigma_s) \, ds = \sum_x \int_0^t \mathbb{E}\mathbb{1}\{\sigma_s(x) = \dots = \sigma_s(y-1)\} \, ds$$

where in both cases the change of order of integral, summation and expectation is justified by positivity of the integrands. Hence the lemma follows from the next claim.  $\Box$ 

Claim 5.3. 
$$\mathbb{E}Q(x,y,t) \leq \mathbb{E}\int_0^t \mathbb{1}\{\sigma_s(x) = \cdots = \sigma_s(y-1)\} ds$$
.

*Proof.* Denote the integrand on the right hand side by  $\chi(s)$ . By the definition of a Toom process,  $\sigma_s|_{[x,y]}$  changes only finitely many times in the time interval [0,t], almost surely. Hence  $\chi$  is Riemann integrable and

$$\int_0^t \chi(s) \, ds = \lim_{\epsilon \to 0} \epsilon \sum_{i=0}^{\lfloor t/\epsilon \rfloor} \chi(\epsilon i).$$

Next let  $t_1 < \cdots < t_k$  be the Poisson arrivals at x in the time interval [0,t] and let

$$B(\epsilon) = \left| \left\{ i : \exists s \in (t_i - \epsilon, t_i) \text{ such that } \sigma_{s^-}|_{[x,y]} \neq \sigma_s|_{[x,y]} \right\} \right|.$$

The condition  $\sigma_s|_{[x,y]}$  changes only finitely many times in the time interval [0,t] also implies

$$\lim_{\epsilon \to 0} B(\epsilon) = 0 \quad \text{almost surely.}$$

Hence by dominated convergence (B is bounded by the number of Poisson arrivals at x),  $\lim_{\epsilon \to 0} \mathbb{E}(B(\epsilon)) = 0$ . Defining

$$\widetilde{Q} = |\{i : \chi(\epsilon i) = 1 \text{ and } \exists t_j \in [\epsilon i, \epsilon(i+1))\}|$$

we get  $Q(x, y, t) \leq \widetilde{Q} + B$  (recall that Q requires also the condition  $\sigma_s(y) = -\sigma_s(x)$ , which is neglected in  $\widetilde{Q}$ ). Since a Toom process has the property that  $\sigma_{t^-}$  is independent of Poisson arrivals at  $[t, \infty)$  we get

$$\mathbb{E}\widetilde{Q} = \mathbb{E}\left[\sum_{i=0}^{\lfloor t/\epsilon\rfloor} \chi(\epsilon i) \mathbb{1}\{\exists t_j \in [\epsilon i, \epsilon(i+1))\}\right] = \mathbb{E}\left[\sum_{i=0}^{\lfloor t/\epsilon\rfloor} \chi(\epsilon i)\right] (\epsilon + O(\epsilon^2)).$$

Since  $\mathbb{E}(\epsilon \sum_{i=0}^{\lfloor t/\epsilon \rfloor} \chi(\epsilon i) - \int_0^t \chi(s) \, ds) \to 0$  (again dominated convergence), the claim follows.

Proof of Lemma 2.11, averaged version (4). Let x < y with y in the support of f, and let  $\epsilon > 0$  and assume for convenience also  $\epsilon < \frac{1}{4}$ . We wish to define the effect of Toom jumps from x to y on  $f(\sigma_{\epsilon})$ . Therefore let  $t_1, \ldots, t_k$  be the times of the Poisson arrivals at x in the time interval  $[0, \epsilon]$  and define, for x < y,

$$D(x,y,\epsilon) = \sum_{i=1}^k (f(\sigma_{t_i}) - f(\sigma_{t_i})) \mathbb{1} \{ \sigma_s(x) = \dots = \sigma_s(y-1) \neq \sigma_s(y) \}.$$

Since, for all z, a Toom process changes at z only finitely many times in any time interval and f is local, we get  $f(\sigma_{\epsilon}) - f(\sigma_{0}) = \sum_{x < y} D(x, y, \epsilon)$ . From this we conclude

$$\mathbb{E}(f(\sigma_{\epsilon}) - f(\sigma_{0})) = \sum_{x < y} \mathbb{E}(D(x, y, \epsilon))$$
(21)

where the exchange of sum and expectation is justified as follows: Let  $Q(y, \epsilon)$  be the number of times  $\sigma_y$  changes sign in the time interval  $[0, \epsilon]$ . Then

$$\sum_{x < y} D(x, y, \epsilon) \le 2||f||_{\infty} \sum_{z \in \text{Supp}(f)} Q(z, \epsilon)$$

which is integrable by Lemma 5.2 and the fact that the sum over the z is finite. Using dominated convergence gives (21).

Moving to the behavior as  $\epsilon \to 0$ , let us first show that

$$\left. \frac{d}{dt} \mathbb{E}(D(x, y, t)) \right|_{t=0} = \mathbb{E}(\mathcal{L}_{x, y} f). \qquad \forall x, y$$

(where the expectation on the right is with respect to  $\sigma_0$ ). To see this let  $E(x, y, \epsilon)$  be the event that for some  $z \in [x, y]$  and some  $t \in [0, \epsilon]$ ,  $\sigma_t(z) \neq \sigma_0(z)$ .

Claim 5.4. 
$$|\mathbb{E}(D(x,y,\epsilon)) - \epsilon \mathbb{E}(\mathcal{L}_{x,y}f)| \le ||f||_{\infty} (6\epsilon \mathbb{P}(E) + \epsilon^2).$$

*Proof.* This is claim is justified by playing around with definitions, but let us do it in

detail nonetheless. Let  $E_1 \subset E$  be the event that for some  $z \in [x, y]$  and some  $t \in [0, \epsilon]$ ,  $\sigma_t(z) \neq \sigma_0(z)$ , and in addition there was no Poisson arrival at x during the time interval [0, t]. Then

$$\mathbb{E}(D \cdot \mathbb{1}\{E_1\}) \le 2||f||_{\infty} \epsilon \mathbb{P}(E_1) \le 2||f||_{\infty} \epsilon \mathbb{P}(E). \tag{22}$$

The first inequality is due to the fact that after  $E_1$  happens, it is still necessary to have a Poisson arrival in  $[t, \epsilon]$ , which has probability less than  $\epsilon$ , independently of  $E_1$ . Even if these events both occur, the maximum effect on the value of f is  $2||f||_{\infty}$ .

Another case which is easily dispensed with is the event that there are two or more arrivals at x (denote the number of arrivals by k and this event by  $E_2$ , so  $E_2 = \{k \geq 2\}$ ). Then  $\mathbb{P}(E_2) < \epsilon \mathbb{P}(E)$  because after the first arrival (which changes  $\sigma_x$  hence is included in E) we still need another arrival, independently. Hence  $\mathbb{E}(D \cdot \mathbb{1}\{E_2\}) < 2||f||_{\infty} \epsilon \mathbb{P}(E)$ .

For the remainder (denote it by  $E_3 = E \cap (E_1 \cup E_2)^c$ ), let

$$G = (\mathcal{L}_{x,y}f)(\sigma_0)\mathbb{1}\{k=1\}.$$

Then

$$\mathbb{E}(D \cdot \mathbb{1}\{E_3\}) = \mathbb{E}(G \cdot \mathbb{1}\{E_3\})$$

because under  $E_3$  these are exactly the same variables. Hence

$$|\mathbb{E}(D \cdot \mathbb{I}\{E_3\}) - \mathbb{E}(G)| \le 2||f||_{\infty} \mathbb{P}(\{k=1\} \setminus E_3) = 2||f||_{\infty} \mathbb{P}(\{k=1\} \cap E_1) \le 2||f||_{\infty} \epsilon \mathbb{P}(E)$$

where the equality follows because  $\{k=1\} \subset E \setminus E_2$ , and the last inequality is as in (22). Finally the definition of  $\mathcal{L}_{x,y}$  gives  $\mathbb{E}(G) = \epsilon e^{-\epsilon} \mathbb{E} \mathcal{L}_{x,y} f$  so  $|\mathbb{E}(G) - \epsilon \mathbb{E} \mathcal{L}_{x,y} f| \leq \epsilon^2$  (recall that we assumed  $\epsilon < \frac{1}{4}$ ). Putting everything together the claim is proved.

Differentiability of  $\mathbb{E}D(x,y,t)$  is now immediate; we write

$$E = \left\{ \sum_{z=x}^{y-1} Q(z, \epsilon) > 0 \right\}$$

and get from Lemma 5.2 and Markov's inequality that  $\mathbb{P}(E) \leq \epsilon C(x,y)$ . Claim 5.4 now gives that  $\frac{d}{dt}\mathbb{E}D(x,y,t) = \mathbb{E}\mathscr{L}_{x,y}f$ .

To be able to sum the derivatives over x we use the assumption that  $\mathbb{E}(l_x(\sigma_0)^{1+\eta}) < \infty$  from the local Condition A and Markov's inequality. We conclude that for any  $\delta > 0$  and  $y \in \mathbb{Z}$  there exists N = N(y) such that

$$\mathbb{E}(l_y(\sigma_t)\mathbb{1}\{l_y(\sigma_t) > N\}) \le \delta \qquad \forall t \in [0, t_0].$$
(23)

Assume also  $N > \frac{1}{\delta}$ . Applying claim 5.3 to all x < y - N and summing gives

$$\sum_{x < y - N} \mathbb{E}D(x, y, \epsilon) \le 2||f||_{\infty} \int_{0}^{\epsilon} l_{y}(\sigma_{t}) \mathbb{1}\{l_{y}(\sigma_{t}) > N\} dt \le 2||f||_{\infty} \cdot \epsilon \delta$$

SO

$$\begin{split} & \overline{\lim}_{\epsilon \to 0} \frac{1}{\epsilon} \sum_{x < y} D(x, y, \epsilon) \leq \sum_{y, x < y - N(y)} \mathbb{E} \mathscr{L}_{x, y} f + 2||f||_{\infty} \delta \\ & \underline{\lim}_{\epsilon \to 0} \frac{1}{\epsilon} \sum_{x < y} D(x, y, \epsilon) \geq \sum_{y, x \in [y - N(y), y)} \mathbb{E} \mathscr{L}_{x, y} f - 2||f||_{\infty} \delta. \end{split}$$

Taking  $\delta \to 0$  gives the lemma.

**Remark 5.5.** The proof of (4) just given can be strengthened slightly when process is stationary. In this case it is enough to assume  $\mathbb{E}(l_x(\sigma(0))) < C(x)$  for all x i.e. it is not necessary to have  $1 + \eta$  moments, 1 is enough. The only difference in the proof is the justification of (23).

Let us now give the proof of the conditional version of the last lemma.

Proof of Lemma 2.11, (3). Let x < y with y in the support of f, and let  $\epsilon \in (0, \frac{1}{4})$ . We wish to define the effect of Toom jumps from x to y on  $f(\sigma_{\epsilon})$ . Therefore let  $t_1, \ldots, t_k$  be the times of the Poisson arrivals at x in the time interval  $[0, \epsilon]$  and define, for x < y,

$$D(x,y,\epsilon) = \sum_{i=1}^k (f(\sigma_{t_i}) - f(\sigma_{t_i^-})) \mathbb{1} \{ \sigma_s(x) = \dots = \sigma_s(y-1) \neq \sigma_s(y) \}.$$

Since, for all z, a Toom process changes at z only finitely many times in any time interval and f is local, we get  $f(\sigma_{\epsilon}) - f(\sigma_{0}) = \sum_{x < y} D(x, y, \epsilon)$ . Let  $\xi(\sigma)$  be the event that  $\sigma(x) = \sigma(x+1) = \cdots = \sigma(y-1)$ . We define two "bad" events,  $B^{1}$  and  $B^{2}$  as follows.

- 1. For  $z \in [x,y]$  let  $B^1(x,z,y,t)$  be the event and that for some s < t we have  $\sigma_s(z) \neq \sigma_0(z)$ ; and that for some  $u \in (s,t)$  there was a Poisson arrival at x and  $\xi(\sigma_{u^-})$  occurred.
- 2. Let  $B^2(x, y, t)$  be the event that  $\xi(\sigma_0)$  occurred, that for some s < t we have  $\sigma_s(x) \neq \sigma_0(x)$ , and that there is a Poisson arrival at x in the time interval (s, t).

Let

$$B^{1}(t) = \bigcup_{\substack{x \le z \le y: \\ y \in \text{Supp}(f)}} B^{1}(x, z, y, t)$$

$$B^{2}(t) = \bigcup_{\substack{x < y: \\ y \in \text{Supp}(f)}} B^{2}(x, y, t)$$

$$B(t) = B^{1}(t) \cup B^{2}(t).$$

Claim 5.6. Almost surely,

$$\sup_{t \in [0,\epsilon]} |\mathbb{E}(f(\sigma_t) - f(\sigma_0) | \sigma_0) - t(\mathcal{L}f)(\sigma_0)|$$

$$\leq 2||f||_{\infty} (\mathbb{P}(B(\epsilon) | \sigma_0) + \epsilon^2) \left(1 + \sum_{y \in \text{Supp}(f)} l_y(\sigma_0)\right).$$

*Proof.* Fix one y in the support of f and one x < y. Define the variable  $G(x, y, t) = (\mathcal{L}_{x,y}f)(\sigma_0) \cdot \mathbb{1}\{k \ge 1\}$  where k is the number of Poisson arrivals at x in the time interval [0,t] (G depends on t only via k). We claim that if  $B(\epsilon)$  doesn't happen then D(x,y,t) = G(x,y,t) for all  $t \in [0,\epsilon]$ .

Indeed, if k=0 then they are both 0, and if  $k \geq 1$  then  $G=(\mathcal{L}_{x,y}f)(\sigma_0)$  while  $D=\sum_i(\mathcal{L}_{x,y}f)(\sigma_{t_i^-})$ , where, as usual,  $t_i$  are the Poisson arrivals at x in the time interval [0,t]. If  $B^1$  doesn't occur, only the first term in this sum may be non-zero (this first arrival will change the value of  $\sigma(x)$ , so any further arrival at x, if it contributes to D it must also trigger  $B^1(x,x,y,\epsilon)$ ). Further, the fact that  $B^1$  doesn't occur implies that

$$(\mathscr{L}_{x,y}f)(\sigma_{t_1^-}) \neq 0 \implies \sigma_{t_1^-}\Big|_{[x,y]} = \sigma_0|_{[x,y]}$$

while the fact that  $B^2$  doesn't occur implies that

$$(\mathscr{L}_{x,y}f)(\sigma_0) \neq 0 \implies \sigma_{t_1^-}\Big|_{[x,y]} = \sigma_0\Big|_{[x,y]}.$$

Thus if neither occurred,  $(\mathscr{L}_{x,y}f)(\sigma_{t_1^-}) = (\mathscr{L}_{x,y}f)(\sigma_0)$  and D = G.

Summing over x and y gives, assuming  $B = B(\epsilon)$  doesn't occur, that

$$f(\sigma_t) - f(\sigma_0) = G(t) := \sum_{x,y} G(x,y,t)$$

(note that the sum defining G is in fact finite). Taking conditional expectation gives

$$\mathbb{E}((f(\sigma_t) - f(\sigma_0)) \cdot \mathbb{1}\{B^c\} \mid \sigma_0) = \mathbb{E}(G \cdot \mathbb{1}\{B^c\} \mid \sigma_0) \tag{24}$$

One remainder can be estimated simply by

$$\mathbb{E}(|f(\sigma_t) - f(\sigma_0)| \cdot \mathbb{1}\{B\} | \sigma_0) \le 2||f||_{\infty} \mathbb{P}(B | \sigma_0)$$
(25)

while for the other we ignore the condition  $\{k \geq 1\}$  in the definition of G and get

$$\mathbb{E}(|G| \cdot \mathbb{1}\{B\} \mid \sigma_0) \leq \mathbb{E}\left(\sum_{x,y} |\mathcal{L}_{x,y} f(\sigma_0)| \cdot \mathbb{1}\{B\} \mid \sigma_0\right)$$

$$= \sum_{x,y} |\mathcal{L}_{x,y} f(\sigma_0)| \cdot \mathbb{P}(B \mid \sigma_0) \leq 2||f||_{\infty} \sum_{y} l_y(\sigma_0) \mathbb{P}(B \mid \sigma_0) \quad (26)$$

Summing (24), (25) and (26) gives

$$|\mathbb{E}(f(\sigma_t) - f(\sigma_0) \mid \sigma_0) - \mathbb{E}(G \mid \sigma_0)| \le 2||f||_{\infty} \mathbb{P}(B \mid \sigma_0) \left(1 + \sum_{y} l_y(\sigma_0)\right)$$

Finally a Poisson process calculation gives  $\mathbb{E}(G(x,y,t) | \sigma_0) = (1-e^{-t})\mathcal{L}_{x,y}f(\sigma_0)$  so  $|\mathbb{E}(G | \sigma_0) - t(\mathcal{L}f)(\sigma_0)| \leq 2||f||_{\infty}\epsilon^2 \sum_y l_y(\sigma_0)$ . Putting everything together the claim is proved.

Thus the lemma will be proved once we estimate  $\mathbb{P}(B)$ . We start with  $B^1$ .

Claim 5.7. With probability 1,

$$\lim_{n \to \infty} 2^n \mathbb{P}(B^1(2^{-n}) \,|\, \sigma_0) = 0.$$

*Proof.* Fix  $x \leq z \leq y$ . We note two estimates for  $B^1(x, z, y, \epsilon)$ . First, ignoring the requirement at z gives

$$\mathbb{P}(B^{1}(x, z, y, \epsilon)) \le \epsilon \sup_{t \in [0, \epsilon]} \mathbb{P}(l_{y} > y - x) \le C(y) \cdot \epsilon(y - x)^{-1 - \eta}$$

where the second inequality is due to our moment assumption on  $\sigma$  and Markov's inequality. Second, ignoring the requirement that the arrival at x actually changes y, and using only the fact that an arrival happened we get from Lemma 5.2

$$\mathbb{P}(B^1(x, z, y, \epsilon)) \le C(z) \cdot \epsilon^2. \tag{27}$$

Summing over all x gives

$$\mathbb{P}\Big(\bigcup_{x} B^{1}(x,z,y,\epsilon)\Big) \leq \epsilon \sum_{x} \min(C(y)(y-x)^{-1-\eta},C(z)\epsilon) \leq C(y,z)\epsilon^{1+\eta/(1+\eta)}$$

where C(y,z) is some constant which depends on y and z but not on  $\epsilon$ . Let us denote

 $B^1(z,y,\epsilon) = \bigcup_x B^1(x,z,y,\epsilon)$ . Markov's inequality now gives for the conditioned events that

$$\mathbb{P}(\mathbb{P}(B^1(z,y,\epsilon) \mid \sigma_0) > \epsilon^{1+\eta/2(1+\eta)}) < C(y,z)\epsilon^{\eta/2(1+\eta)}.$$

This implies, by the Borel-Cantelli Lemma, that with probability 1,

$$\lim_{n \to \infty} 2^n \mathbb{P}(B^1(z, y, 2^{-n}) \mid \sigma_0) = 0$$
 (28)

for all z and y.

There is another important consequence of the conditioning over  $\sigma_0$ . Fix some value of  $\sigma_0$  and let  $z^* = z^*(y) = y - l_y(\sigma_0) - 1$ . For any  $x \leq z^*$  we have that

$$\bigcup_{z=x}^{y} B^{1}(x,z,y,\epsilon) \subset B^{1}(x,z^{*},y,\epsilon)$$

because the requirement that at the time u (u from the definition of  $B^1$ ) we have the event that  $\xi(\sigma_{u^-})$  occurs is fulfilled only if  $\sigma_{u^-}(z^*) \neq \sigma_0(z^*)$ . Thus

$$\overline{\lim}_{n\to\infty} 2^n \mathbb{P}\Big(\bigcup_{x\leq z^*} \bigcup_z B^1(x,z,y,2^{-n}) \,\Big|\, \sigma_0\Big) \leq \overline{\lim}_{n\to\infty} 2^n \mathbb{P}(B^1(z^*,y,2^{-n}) \,|\, \sigma_0) = 0.$$

The remaining x (i.e.  $x > z^*$ ) can be estimated directly by summing (28) over z from  $z^* + 1$  to y. We get

$$2^{n} \mathbb{P}\left(B^{1}(2^{-n}) \mid \sigma_{0}\right) \leq 2^{n} \sum_{y \in \text{Supp } f} \sum_{z=z^{*}(y)}^{y} \mathbb{P}(B^{1}(z, y, 2^{-n}) \mid \sigma_{0}) \to 0$$

proving the claim.

The estimate of  $B^2(\epsilon)$  is much simpler and we will not dignify it with a claim. By Lemma 5.2,

$$\mathbb{P}(B^2 \mid \sigma_0) \le \epsilon^2 \sum_{y \in \text{Supp } f} \sum_{x=y-l_y(\sigma_0)}^y C(x)$$

and hence

$$\lim_{n \to \infty} 2^n \mathbb{P}(B^2(2^{-n}) \mid \sigma_0 = 0$$
 (29)

almost surely.

The lemma is now proved. Applying claim 5.6 for  $t \in [2^{-n-1}, 2^{-n}]$  we get

$$\left| \frac{1}{t} \mathbb{E}(f(\sigma_t) - f(\sigma_0) | \sigma_0) - \mathcal{L}f(\sigma_0) \right| \\
\leq 2||f||_{\infty} (2^{n+1} \mathbb{P}(B(2^{-n}) | \sigma_0) + 2^{-n+1}) \left( 1 + \sum_{y} l_y(\sigma_0) \right).$$

By claim 5.7 and (29),  $2^n \mathbb{P}(B(2^{-n}) | \sigma_0) \to 0$  almost surely, and the other terms on the right hand side are constant for any fixed  $\sigma_0$ .

## References

- [1] R. Alexander. Time evolution for infinitely many hard spheres. Commun. Math. Phys., 49(3):217–232, 1976. springer.com/BF01608728.
- [2] Arvind Ayyer, Anne Schilling, Benjamin Steinberg, and Nicolas M. Thiéry. Markov chains,  $\mathscr{R}$ -trivial monoids and representation theory. *Internat. J. Algebra Comput.*, 25(1-2):169-231, 2015. worldscientific.com/doi/10.1142/S0218196715400081.
- [3] G.T. Barkema, P.L. Ferrari, J.L. Lebowitz, and H. Spohn. Kardar-parisi-zhang universality class and the anchored toom interface. *Phys. Rev. E*, 90(4):042116, Oct 2014. aps.org/pre/abstract/10.1103/PhysRevE.90.042116.
- [4] Nick Crawford and Woijcech De Roeck. Clt pushed tagged particles. in preparation, 2015.
- [5] B. Derrida, J. L. Lebowitz, E. R Speer, and H. Spohn. Dynamics of an anchored Toom interface. *Journal of Physics A: Mathematical and General*, 24(20):4805, 1991. iop.org/0305-4470.
- [6] P. Devillard and H. Spohn. Universality class of interface growth with reflection symmetry. *Journal of Statistical Physics*, 66(3-4):1089–1099, 1992. springer.com.
- [7] J Krug and H Spohn. Kinetic roughening of growing surfaces. In C. Godrèche, editor, *Solids far from equilibrium*, *Vol. 1*. Cambridge University Press, Cambridge, 1991.
- [8] T. M. Liggett. Long range exclusion processes. Ann. Probab., 8(5):861–889, 1980.jstor.org/2242933.
- [9] T. M. Liggett. Interacting Particle Systems, volume 276. Springer, 1985.
- [10] László Lovász and Peter Winkler. Mixing times. In Microsurveys in discrete probability (Princeton, NJ, 1997), volume 41 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci., pages 85–133. Amer. Math. Soc., Providence, RI, 1998.

- [11] C. Maes, F. Redig, E. Saada, and A. Van Moffaert. On the thermodynamic limit for a one-dimensional sandpile process. *Markov processes and related fields*, 6:1681–1698, 1998. arXiv:9810093.
- [12] M. Paczuski, M. Barma, S. N. Majumdar, and T. Hwa. Fluctuations of a nonequilibrium interface. *Phys. Rev. Lett.*, 69:2735–2735, Nov 1992. aps.org/10.1103.
- [13] A.L. Toom. Stable and attractive trajectories in multicomponent systems. *Advances in Probability*, 6(1):549–575, 1980.

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